Time Series Analysis: Exercises 2

1. a) Show that the processes Z_t and Z_t' defined by

$$Z_t = W_t + \theta W_{t-1}, \qquad W_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2),$$
$$Z_t = W_t + \theta^{-1} W_{t-1}, \qquad W_t \overset{iid}{\sim} \mathcal{N}(0, \theta^2\sigma^2),$$

have identical second-order properties. Which, if either, is the more appropriate for an ARMA process?

   b) If $W_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$, is the process

$$Y_t = 0.5Y_{t-1} + 0.5W_t + 0.75W_{t-1} - 0.5W_{t-2}, \qquad t \in \mathbb{Z},$$

a well-defined ARMA model? If not, reduce it to the best form, using (a) or otherwise.

2. Consider a stationary ARMA model with a constant, i.e.,

$$\phi(B)Y_t = \alpha + \theta(B)W_t,$$

where W_t is a white noise. Show that if there is no autoregressive part, then $\mathrm{E}(Y_t) = \alpha$, and find $\mathrm{E}(Y_t)$ in the general case. Discuss the interpretation of α for an ARIMA$(p, 1, q)$ model with a constant,

$$\phi(B)DY_t = \alpha + \theta(B)W_t.$$

3. Consider an AR(1) model with $|\phi| < 1$. Find the general form of the h-step-ahead predictor $\hat{Y}_{n+h}$ based on data $Y_1, \ldots, Y_n$ and show that

$$\mathrm{E}\left\{(\hat{\mathrm{Y}}_{n+h} - \mathrm{Y}_{n+h})^2\right\} = \sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2}.$$

As $h \to \infty$ sketch the behaviour of $\hat{Y}_{n+h}$ and its variance. Discuss.

4. a) Find the covariance function corresponding to the spectral density $f(\omega) = \sin(2\pi\omega)$, $0 < \omega < 1/2$.

   b) Do the same with $f(\omega) = \mathrm{I}_{(0 \le \omega \le 1/4)} - \mathrm{I}_{(1/4 \le \omega \le 1/2)}$, where I_A denotes the indicator of the set A. Discuss.

5. Suppose that Y_t follows an AR(2) process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + W_t,$$

where W_t is a white noise process with variance σ^2. Use the fact that the spectral densities are related by $|\phi(\omega)|^2 f_Y(\omega) = f_W(\omega)$ to show that

$$f_Y(\omega) = \frac{\sigma^2}{1 + \phi_1^2 + \phi_2^2 + 2\phi_1(\phi_2 - 1)\cos(2\pi\omega) - 2\phi_2\cos(4\pi\omega)}, \quad 0 < \omega < 1/2.$$

6. A harmonic process may be defined as

$$X_t = \sum_{j=1}^{K} A_j \cos(2\pi\omega_j t + 2\pi U_j),$$

where $A_1, \ldots, A_K$ and K are constants and $U_1, \ldots, U_K \stackrel{iid}{\sim} U(0,1)$. Find its mean and autocorrelation function, and deduce that the process is stationary.

7. Show that the periodogram ordinate $I(\omega)$ may be written as

$$n^{-1}\left[\left\{\sum_{t=1}^{n}(y_t - \bar{y})\sin(\omega t)\right\}^2 + \left\{\sum_{t=1}^{n}(y_t - \bar{y})\cos(\omega t)\right\}^2\right],$$

and by squaring out the sums and using the facts that $\cos(a)^2 + \sin(a)^2 = 1$ and $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a - b)$, deduce that

$$nI(\omega) = \sum(y_t - \bar{y})^2 + 2\sum_{k=1}^{n-1}\sum_{t=k+1}^{n}(y_t - \bar{y})(y_{t-k} - \bar{y})\cos(k\omega),$$

and hence that

$$I(\omega) = \hat{\gamma}(0) + 2\sum_{k=1}^{n-1}\hat{\gamma}(k)\cos(k\omega).$$

Hence show (**advanced question**) that the normalized periodogram $I(\omega)/\hat{\gamma}(0)$ is the discrete Fourier transform of the correlogram.

8. If X_t and Y_t are independent and stationary zero-mean time series, show that the spectral density for the product series $Z_t = X_t Y_t$ can be written as

$$f_Z(\omega) = \int_{-1/2}^{1/2} f_X(\omega - \nu) f_Y(\nu) d\nu.$$

9. Aliasing arises when a time series is subsampled. For a simple example of this, let X_t be an AR(1) time series with a $\gamma(t) = \sigma^2\rho^{-|t|}$ for $t \in \mathbb{Z}$ and $|\rho| < 1$ and let Y_t denote the series such that $Y_t = X_{kt}$ for some $k = 2, 3, \ldots$. Compute the corresponding spectra $f_X(\omega)$ and $f_Y(\omega)$ and sketch them in case $k = 2$ and $\rho = -0.9$. Discuss.

10. Consider the following representation of the periodogram:

$$I(\omega) = n^{-1}|d(\omega)|^2, \qquad d(\omega) = \sum_{t=0}^{n-1} y_t \exp(it\omega),$$

with ω of the form $\omega_j = 2\pi j/n$ for some j an integer between 0 and $n - 1$. Show that one can write

$$d(\omega_j) = \sum_{t_0=0}^{r-1} S(t_0, j_0) \exp(2\pi i j t_0/n).$$

Hint: suppose n factorizes as $n = rs$.

11. Data $y_1, \ldots, y_n$ are available from a time series thought to have structure

$$Y_t = \theta \sin\{2\pi(wt + \beta)\} + W_t, \qquad t = 0, \pm 1, \pm 2, \ldots,$$

where W_t is white noise, and θ, ω and β are unknown.

a) Compute the spectrum of Y_t, and hence describe the periodogram $I_1, \ldots, I_m$ of $y_1, \ldots, y_n$.
b) It is desired to test the null hypothesis $\theta = 0$ against the alternative $\theta \neq 0$. Explain why you would expect the null hypothesis for large values of the statistic $T = \max\{I_1, \ldots, I_m\}$.
c) If n is odd and large, show that when $\theta = 0$,

$$\mathrm{P}(T \leq t) = \left\{1 - e^{-t/\sigma^2}\right\}^m = \exp\left\{-\exp\left\{-t/\sigma^2 + \log m\right\}\right\},$$

and show that the maximum likelihood estimator of σ^2 is $\hat{\sigma}^2 = m^{-1}(I_1 + \ldots + I_m)$.
d) Hence establish that a test a test size α rejects the hypothesis $\theta = 0$ when

$$\max\{I_1, \ldots, I_m\}/\hat{I} > -\log(1 - \alpha^{1/m}).$$

12. Consider the local average around the Fourier frequency ω_i,

$$T = \frac{1}{2p+1} \sum_{j=-p}^{p} I(2\pi\omega_{i+j}),$$

of the periodogram ordinates of a time series with a smooth spectrum $f(\omega)$. By writing

$$I(2\pi\omega_{i+j}) = f(\omega_{i+j})E_j = E_j\left\{f(\omega_i) + \delta f'(\omega_i) + \frac{1}{2}\delta^2 f''(\omega_i) + \ldots\right\},$$

where $E_{-p}, \ldots, E_p \overset{iid}{\sim} Exp(1)$ and $\delta = O(1/n)$, establish that

$$T = f(\omega_i)\hat{E} + \delta f'(\omega_i)A + \frac{1}{2}\delta^2 f''(\omega_i)B,$$

where A and B are random variables that you should find. Hence deduce that if p is fixed and $n \to \infty$,

$$T \sim f(\omega_i)\frac{1}{2(2p+1)}\chi^2_{2(2p+1)}.$$

Hence find the large-sample mean and variance of T.

13. Derive the relation between G_k and $G_p(\nu)$ at the bottom of the page 40 of the lectures notes on Chapter 3 about spectral methods.