Time Series Analysis: Exercise 1

November 11, 2022

1. Consider a white noise process $W_t \sim \text{WN}(0, \sigma^2), t > 0$, for which we define a process X_t, $t \geq 1$ as
$$X_t = \max\{W_t, W_{t-1}\}.$$
   (a) Is this process weakly stationary?

   (b) Is it strictly stationary?

2. Let $X_t = a\cos(At + V)$, where a is a constant and $A \sim F$ for some distribution F, and $V \sim U(0, 2\pi)$ (uniform distribution on $[0, 2\pi]$) are independent random variables. Explain why we can suppose that $0 \leq A \leq 2\pi$, and find the mean and covariance of the process X_t. Is it weakly stationary? Would it change the distribution if the distribution of V was not uniform?

3. Consider a white noise process $W_t \sim \text{WN}(0, \sigma^2), t > 0$, for which we define two processes X_t and Y_t, $t \geq 1$ as
$$X_t = \frac{1}{2}(W_t + W_{t-1}),$$
and
$$Y_t = \frac{1}{3}(W_t + W_{t-1}).$$
   (a) What is $\text{Cov}(X_t, Y_{t+1})$?

   (b) Compute the cross-covariance function $\gamma_{X,Y}(s, t)$.

4. (a) Show that the covariance and autocorrelation functions, γ and ρ, of a weakly stationary time series are symmetric around zero, that is, $\gamma(h) = \gamma(-h)$ and $\rho(h) = \rho(-h)$, for all lags h.

   (b) Consider two weakly stationary series X_t and Y_t and define the cross-covariance function as $\gamma_{X,Y}(h) = \text{Cov}(X_{t+h}, Y_t)$. Is this also symmetric around zero?

5. For a strictly stationary process X_t, $t \geq 1$, indicate for each statement if it is true or false.

   (a) for $t, s > 0$, then the bivariate vectors (X_s, X_t) and (X_t, X_s) have the same distribution.

   (b) for $t, s > 0$, then $\text{Cov}(X_t, X_s) = \text{Cov}(X_s, X_t)$.

   (c) for $t, s > 0$, then $E(X_1 X_{1+s}) = E(X_t X_{t+s})$.

(d) for $t, s, h > 0$, then the bivariate vectors (X_t, X_{s+h}) and (X_1, X_h) have the same distribution.

6. Given a Gaussian white noise process $W_t \sim \text{WN}(0, \sigma^2)$, $t \geq 0$ and another (not Gaussian) white nose process $Z_t \sim \text{WN}(0, \sigma^2)$, $t \geq 0$, for each of the following processes $X_t, t \geq 1$, indicate whether it is strictly stationary, only weakly stationary or neither of the two.

   (a) $X_t = 2X_{t-1} + W_t$.

   (b) $X_t = 2(W_t + W_{t-1})$.

   (c) $X_t = W_{\lfloor \frac{(t-1)}{2} \rfloor}$.

   (d) $X_t = \begin{cases} W_1 & \text{for t odd} \\ W_2 & \text{for t even} \end{cases}$.

   (e) $X_t = (-1)^t W_t$.

   (f) $X_t = \begin{cases} W_t & \text{for t odd} \\ Z_t & \text{for t even} \end{cases}$.

7. Recall the backshift operator B defined by $BX_t = X_{t-1}$ for any time series X_t. Let $W_t \sim \text{WN}(0, \sigma^2)$, $t \geq 0$ be a white noise process.

   (a) Express $(1 - B)^2 X_t$.

   (b) Compute $(1 - B)^2 X_t$ for the time series $X_t = t^2 + W_t$.

   (c) What would be the coefficient multiplying X_{t-2} in the expression of $(1 - B)^n X_t$, for $n > 0$?

   (d) Show that $(1 + B)(1 - B) = 1 - B^2$.

8. Let $W_t \sim \text{WN}(0, \sigma^2)$, $t \geq 0$ be a white noise process, we define the AR(1) process $X_t = 0.9X_{t-1} + W_t$ for $t \geq 1$.

   (a) Compute $\text{Cov}((1 - B)X_t, X_t)$

   (b) Compute $\text{Cov}((1 - B)X_t, BX_t)$.

9. Let $S_t = \frac{1}{2}(Y_t + Y_{t-1})$ be a moving average process, Express the residual series

$$Y_t - S_t,$$

   in terms of the backshift operator B.

10. Let $Y_t = p(t) + X_t$, where X_t is a weakly stationary series with covariance function γ.

   (a) If $p(t) = \beta_0 + \beta_1 t$, show that the differenced series $(1 - B)Y_t$ is stationary and find its covariance function.

   (b) If $p(t) = \sum_{r=0}^{k} \beta_r t^r$, show that $(1 - B)Y_t$ has a polynomial trend of degree at most $k - 1$. You should then deduce that $(1 - B)^k Y_t$ is stationary.

   (c) If X_t is a white noise process, show that the series $(1 - B)^k Y_t$is a moving average.

(d) Find the covariance function in b).

11. Let $y_1, \ldots, y_n$ represent the observations of $(Y_t)_{t \in Z}$ at $t = 1, ..., n$ of a discrete time series taking values 0 and 1 in which

$$P(Y_j = y_j | Y_1 = y_1, \ldots, Y_{j-1} = y_{j-1}) = P(Y_j = y_j | Y_{j-1} = y_{j-1}) = \theta_{y_{j-1} y_j}.$$

Note that $\theta_{11} = 1 - \theta_{12}$ and $\theta_{22} = 1 - \theta_{21}$, where θ_{12} and θ_{21} are the transitions probabilities from state 1 to 2 and vice versa.

(a) Show that the likelihood can be written as

$$L(\theta) = \theta_{12}^{n_{12}} (1 - \theta_{12})^{n_{11}} \theta_{21}^{n_{21}} (1 - \theta_{21})^{n_{22}} P(Y_1 = y_1),$$

where n_{xy} is the number of transitions of $x \to y$ in the observed time series $(y_1, \ldots, y_n)$.

(b) Find the maximum likelihood estimates $\hat{\theta}_{12}$ and $\hat{\theta}_{21}$ and their asymptotic variances.

(c) Advanced question: Construct a likelihood ratio test to compare this model with one in which there is no dependence between the successive states (i.e. $\theta_{12} = \theta_{21}$).

(d) Apply the results from b) and c) to the (short) sequence 12111122121212111212212121.

12. (a) Show that the variance of the average $\overline{Y}$ of data $Y_1, \ldots Y_n$ from a stationary process with covariance function γ and correlation function ρ is

$$\operatorname{Var}(\overline{Y}) = \frac{\gamma(0)}{n} \left\{ 1 + \frac{2}{n} \sum_{h=1}^{n-1} (n-h)\rho(h) \right\}.$$

Note that the positive correlation inflates the variance and conversely.

(b) Show that for an AR(1) process with regressive parameter α and $|\alpha| < 1$ for large n,

$$\operatorname{Var}(\overline{Y}) \sim n^{-1} \frac{\sigma^2}{(1-\alpha)^2}.$$

What happens if $\alpha \to 1$, $\alpha \to -1$, $\alpha = 0$?

Hint: if X is a geometric random variable with parameter p, then $P(X = k) = (1-p)^{k-1}p$, then $E(X) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = 1/p$.

(c) Remember from the lecture notes that for a difference-stationary series X_t, the slope for the linear trend can be estimated by the mean of the difference series $Z_t = (1 - B)X_t$. Explain how you would proceed to build an asymptotic confidence interval for this slope if you assume to know the covariance function $\gamma(h)$ of the difference series Z_t.

13. The variogram of a weekly stationary stochastic process Y_t is defined as

$$V(h) = \frac{1}{2} E\left\{ (Y_t - Y_{t+h})^2 \right\}.$$

(a) Show that for a stationary process, $V(h) = \gamma(0)(1 - \rho(h))$ and sketch this for $h \geq 0$ in the two cases of Y being white noise or having correlation function $\rho(h) = \exp(-\lambda|h|)$.

(b) If time series data y_t are available at unequally-spaced times, $t \in \{t_1, \ldots, t_n\}$ explain why a natural way to estimated $V(h)$ for $h \geq 0$ is to plot the quantities $v_{ij} = (y_{t_i} - y_{t_j})^2/2$ against the $|t_i - t_j|$ for all $n(n-1)/2$ pairs of times, and to smooth the resulting set of points.

(c) In most cases there will be several values of v_{ij} for each $|t_i - t_j|$. In this case explain why it is best to replace all the v_{ij} such that $|t_i - t_j| = h$ by their average $\overline{v}_h$.

14. Let $Y_1, Y_2, \ldots, Y_n$ satisfy the one-step Markov property, i.e.

$$P(Y_r = y_r | Y_{r-1} = y_{r-1}, \ldots Y_1 = y_1) = P(Y_r = y_r | Y_{r-1} = y_{r-1}), \text{ for } y_1, \ldots, y_r \in \mathbb{R},$$

for $r > 1$, and suppose for simplicity that the random variables $Y_1, Y_2, \ldots, Y_n$ take discrete values. Show that for $n > 1$,

$$P(Y_1 = y_1, \ldots, Y_n = y_n) = P(Y_1 = y_1) \prod_{r=2}^{n} P(Y_r = y_r | Y_{r-1} = y_{r-1}).$$

15. If $Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t$ where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ iid show that

$$Y_t | Y_{t-1} = y_{t-1}, \ldots, Y_1 = y_1 \quad \sim \quad \mathcal{N}(\mu + \phi_1(y_t - 1 - \mu) + \phi_2(y_{t-2} - \mu), \sigma^2), t = 3, ..., n$$

and hence write down the likelihood for $\mu, \sigma^2, \phi_1, \phi_2$

16. (a) Calculate the roots of the polynomial

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2,$$

for the following different specific values

$$\phi_1 = 0.8, \phi_2 = 0.4$$
$$\phi_1 = 0.6, \phi_2 = -0.1$$
$$\phi_1 = -0.1, \phi_2 = 0.7$$

(b) For a given white-noise process W_t, show whether the AR(2) processes $\Phi(B)X_t = W_t$, with the three possible sets of values from point a) are stationary. If stationary, indicates whether the correlation function $\rho(h)$ will present an oscillating behaviour/shape.

17. In this exercise we introduce the following two-dimensional time series model member of the family of vector-autoregressive models (VAR) where

$$X_t = \Phi X_{t-1} + W_t$$

where

$$X_t = \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}, \; W_t = \begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix},$$

with $W_t^{(1)}$ and $W_t^{(2)}$ independent white-noise processes and Φ a 2×2 matrix given by

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

Provide sufficient conditions for the two-dimensional process X_t to be causal and stationary.