The Lasso identifies under weaker conditions than sparse eigenvalues

$$(*) \quad \phi^2_{min}(m) = \min_{\substack{\beta \neq 0; \\ \|\beta\|_0 \leq m}} \frac{\beta^T \hat{\Sigma} \beta}{\|\beta\|_2^2}$$

place "additional" restrictions in $(*)$:

$$\kappa^2(m, 3) = \min_{\substack{S \subseteq \{1, \ldots, p\} \\ |S| \leq m}} \; \min_{\substack{\beta \neq 0 \\ \underbrace{\|\beta_{S^c}\|_1 \leq 3\|\beta_S\|_1}_{\text{"cone condition"}}}} \frac{\beta^T \hat{\Sigma} \beta}{\|\beta_S\|_2^2}$$

indep. of p

restricted minimal eigenvalue (Bickel, Ritov, Tsybakov 2009)

$$\underset{p \times 1}{(\beta_A)_j} = \begin{cases} \beta_j & j \in A \\ 0 & j \notin A \end{cases}, \quad A \subseteq \{1, \ldots, p\}$$

can show that:

$$\boxed{\begin{array}{l}\text{cone condition is fulfilled for } \hat{\beta}-\beta^0 \text{ on the set } S_0 \\ \|(\hat{\beta}-\beta^0)_{S_0^c}\|_1 \leq \|\hat{\beta}_{S_0^c}\|_1 \leq 3\,\|(\hat{\beta}-\beta^0)_{S_0}\|_1\end{array}}$$

if $\underline{\kappa^2(S_0, 3) > 0}$: can identify with the Lasso

$$\left.\begin{array}{l} X\hat{\beta} = X\beta^0 \Longrightarrow \hat{\beta}_{S_0} = \beta^0_{S_0} \\ \rightsquigarrow \hat{\beta}_{S_0^c} = 0, \quad \beta^0_{S_0^c} \overset{\text{Def}}{=} 0 \end{array}\right\} \Rightarrow \hat{\beta} = \beta^0$$

weaker condition: compatibility condition for the set $\underline{S_0}$

$$\phi_0^2 = \min_{\substack{\beta \neq 0, \\ \|\beta_{S_0^c}\|_1 \leq 3 \|\beta_{S_0}\|_1}} \frac{s_0 \, \beta^T \Sigma \beta}{\cancel{\|\beta_{S_0}\|_2^2} \; \|\beta_{S_0}\|_1^2}$$

Cauchy-Schwarz: $\|\beta_{S_0}\|_1 \leq \sqrt{s_0} \, \|\beta_{S_0}\|_2$

$$\Longrightarrow \phi_0^2 \geq \kappa^2(s_0, 3)$$

compatibility condition is slightly weaker than restricted eigenvalue condition; yet it is still good enough to identify with the Lasso