ARMA Signals

ARMA models can represent spectra with both peaks (AR part) and valleys (MA part).

$$A(q)y(t) = B(q)e(t)$$

$$\phi(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2 = \frac{\sum_{k=-m}^{m} \gamma_k e^{-i\omega k}}{|A(\omega)|^2}$$

where

$$\begin{aligned} \gamma_k &= E\left\{[B(q)e(t)][B(q)e(t-k)]^*\right\} \\ &= E\left\{[A(q)y(t)][A(q)y(t-k)]^*\right\} \\ &= \sum_{j=0}^{n} \sum_{p=0}^{n} a_j a_p^* r(k+p-j) \end{aligned}$$

ARMA Spectrum Estimation

Two Methods:

1. Estimate $\{a_i, b_j, \sigma^2\}$ in $\phi(\omega) = \sigma^2 \left|\frac{B(\omega)}{A(\omega)}\right|^2$

- nonlinear estimation problem; can use an approximate linear *two-stage least squares* method

- $\hat{\phi}(\omega)$ is guaranteed to be ≥ 0

2. Estimate $\{a_i,\ r(k)\}$ in $\phi(\omega) = \frac{\sum_{k=-m}^{m} \gamma_k e^{-i\omega k}}{|A(\omega)|^2}$

- linear estimation problem (the Modified Yule-Walker method).

- $\hat{\phi}(\omega)$ is *not* guaranteed to be ≥ 0

Two-Stage Least-Squares Method

Assumption: The ARMA model is invertible:

$$\begin{aligned} e(t) &= \frac{A(q)}{B(q)} y(t) \\ &= y(t) + \alpha_1 y(t-1) + \alpha_2 y(t-2) + \cdots \\ &= \text{AR}(\infty) \text{ with } |\alpha_k| \to 0 \text{ as } k \to \infty \end{aligned}$$

Step 1: Approximate, for some large K

$$e(t) \simeq y(t) + \alpha_1 y(t-1) + \cdots + \alpha_K y(t-K)$$

1a) Estimate the coefficients $\{\alpha_k\}_{k=1}^{K}$ by using AR modelling techniques.

1b) Estimate the noise sequence

$$\hat{e}(t) = y(t) + \hat{\alpha}_1 y(t-1) + \cdots + \hat{\alpha}_K y(t-K)$$

and its variance

$$\hat{\sigma}^2 = \frac{1}{N-K} \sum_{t=K+1}^{N} |\hat{e}(t)|^2$$

Two-Stage Least-Squares Method, con't

Step 2: Replace $\{e(t)\}$ by $\hat{e}(t)$ in the ARMA equation,

$$A(q)y(t) \simeq B(q)\hat{e}(t)$$

and obtain estimates of $\{a_i,\ b_j\}$ by applying least squares techniques.

Note that the a_i and b_j coefficients enter linearly in the above equation:

$$\begin{aligned} y(t) - \hat{e}(t) &\simeq [-y(t-1) \ldots - y(t-n), \\ &\qquad \hat{e}(t-1) \ldots \hat{e}(t-m)]\theta \\ \theta &= [a_1 \ldots a_n \, b_1 \ldots b_m]^T \end{aligned}$$