Concerns with Singular Value Decomposition in R Software

Using singular value decomposition, any second-order tensor is given as

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}, \tag{1}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the orthogonal tensors, and $\boldsymbol{\Sigma}$ is the diagonal matrix (Eigenvalue matrix). For a symmetric matrix, the orthogonal tensors are the same, i.e., $\mathbf{U}$=$\mathbf{V}$. There is an issue with the singular value decomposition function (svd) in R-software which is illustrated using the following example.

Example:
The symmetric matrix $\mathbf{A}$ is assumed as

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}. \tag{2}$$

Upon finding the singular value decomposition of matrix $\mathbf{A}$ using svd(A) in R software, one arrives at

$$\begin{aligned} \mathbf{U} &= \begin{bmatrix} -0.7071068 & -0.7071068 \\ -0.7071068 & 0.7071068 \end{bmatrix} \\ \boldsymbol{\Sigma} &= \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \\ \mathbf{V} &= \begin{bmatrix} -0.7071068 & 0.7071068 \\ -0.7071068 & -0.7071068 \end{bmatrix} \end{aligned} \tag{3}$$

The tensors $\mathbf{U}$ and $\mathbf{V}$ should be the same for a symmetric matrix. However, the tensors (Eq.(3)) for the chosen symmetric matrix given in Eq.(2) are different. The diagonal matrix should be $\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$, and a minus sign is missing in the obtained diagonal matrix '$\boldsymbol{\Sigma}$'(Eq.(3)). As a result of these issues, a difference is observed in finding the exponential of matrix $\mathbf{A}$. The exponential of the assumed matrix can be obtained using different methods, and comparison of these methods using R software are explained as follows.

Comparison of the exponential of a matrix using singular value decomposition and series expansion:
The series expansion of the exponential of a matrix is

$$\mathrm{e}^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + ..., \tag{4}$$

and using the singular value decomposition of matrix $\mathbf{A}$ (Eq.(1)) in the above series, the exponential of symmetric matrix $\mathbf{A}$ is given through

$$\mathrm{e}^{\mathbf{A}} = \mathbf{U}\mathrm{e}^{\mathbf{\Sigma}}\mathbf{U}^{\mathrm{T}}. \tag{5}$$

Using the singular value decomposition of the assumed matrix $\mathbf{A}$ (Eqs.(3)) in the above equation, the exponential of symmetric matrix $\mathbf{A}$ is given as

$$\mathrm{e}^{\mathbf{A}} = \begin{bmatrix} 84.24935 & 64.16381 \\ 64.16381 & 84.24935 \end{bmatrix}. \tag{6}$$

On using the series expansion(Eq.(4)), the exponential of the assumed matrix(Eq.(2)) is obtained as

$$\mathrm{e}^{\mathbf{A}} = \begin{bmatrix} 74.23147 & 74.18169 \\ 74.18169 & 74.23147 \end{bmatrix} \tag{7}$$

with a convergence of 10^{-6}. The exponential of matrix $\mathbf{A}$ obtained from the series is correct. One arrives at the same components by using the correct eigenvalues in the eigenvalue matrix $\mathbf{\Sigma}$ in Eq.(5), i.e., $\mathbf{\Sigma} = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$

Comparison of the exponential of a matrix using projection tensors and eigen values, and series expansion:
One can obtain the exponential of a matrix using eigenvalues and projection matrices. For any 2x2 matrix, the projection matrices are as follows:

$$\mathbf{P}_1 = \frac{\mathbf{A} - \lambda_1 \mathbf{I}}{\lambda_1 - \lambda_2}$$

and

$$\mathbf{P}_2 = \frac{\mathbf{A} - \lambda_2 \mathbf{I}}{\lambda_2 - \lambda_2}, \tag{8}$$

where λ_1 and λ_2 are eigenvalues. The matrix 'A' is related to eigenvalues and projection tensors as

$$\mathbf{A} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2. \tag{9}$$

Using the above equation in series expansion in Eq.(4), and the relations of projection tensors, i.e.,

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{0},\ \mathbf{P}_1^2 = \mathbf{P}_1,\ \mathbf{P}_2^2 = \mathbf{P}_2 \text{ and } \mathbf{P}_1 + \mathbf{P}_2 = \mathbf{I}, \tag{10}$$

the exponential of a matrix is given through

$$\mathrm{e}^{\mathbf{A}} = \mathrm{e}^{\lambda_1}\mathbf{P}_1 + \mathrm{e}^{\lambda_2}\mathbf{P}_2. \tag{11}$$

Substituting the projection matrices(Eq.(8)) and the corresponding eigenvalues for the assumed matrix(Eq.(2)) in the above equation, and the exponential of a matrix is given as

$$\mathrm{e}^{\mathbf{A}} = \begin{bmatrix} 74.23147 & 74.18169 \\ 74.18169 & 74.23147 \end{bmatrix} \tag{12}$$

The matrices obtained from projection matrices and series expansion are same (see Eqs. (7) and (12)).

Observations:
The expressions for the exponential of a matrix using singular value decomposition and projection tensors are obtained from series expansion. However, numerical inconsistency is observed between the exponential of matrix derived using singular value decomposition and the series form (see Eqs.(6) and (7)), and an accurate components for the exponential of matrix are obtained by using projection tensors(see Eqs.(12) and (7)). Therefore, there is an issue with singular value decomposition function(svd()) used in R software.