# Solution to Series 10

**1. a)** i) $X_t = t + E_t$ is **not stationary**, since the expected value $E[X_t] = E[t + E_t] = t$ is not constant.

ii) For the series $Y_t = X_t - X_{t-1}$ holds

$$Y_t = X_t - X_{t-1} = t + E_t - (t - 1 + E_{t-1}) = 1 + E_t - E_{t-1}\,,$$

this means $Y_t$ is an $MA(1)$-process with $\mu = 1$ and $\beta_1 = -1$, that is **stationary**.

iii) The series $Z_t = X_t - t$ is **stationary**, since $Z_t = X_t - t = t + E_t - t = E_t$ .

**b) Series $Y_t$:** We can calculate the autocovariances:

$$\begin{aligned}\gamma_{11}(k) &= \text{Cov}(Y_t, Y_{t+k}) = \text{Cov}(1 + E_t - E_{t-1}, 1 + E_{t+k} - E_{t+k-1}) \\ &= \text{Cov}(E_t, E_{t+k}) - \text{Cov}(E_t, E_{t+k-1}) - \text{Cov}(E_{t-1}, E_{t+k}) + \text{Cov}(E_{t-1}, E_{t+k-1}) \\ &= \begin{cases} 2\sigma^2 & k = 0 \\ -\sigma^2 & k = \pm 1 \\ 0 & |k| > 1 \end{cases}\end{aligned}$$

Thus we get the autocorrelations:

$$\begin{aligned}\rho_{11}(0) &= 1\,, \\ \rho_{11}(\pm 1) &= \frac{\gamma_{11}(1)}{\gamma_{11}(0)} = -\frac{1}{2}\,, \\ \rho_{11}(k) &= 0\,, \quad \text{for } |k| > 1\,.\end{aligned}$$

**Series $Z_t$:** Since $Z_t = E_t$ is white noise the following holds:
$\gamma_{22}(0) = \sigma^2$ und $\gamma_{22}(k) = 0$, für $|k| \geq 1$,
Thus $\rho_{22}(0) = 1$ und $\rho_{22}(k) = 0$, für $|k| \geq 1$ .

**Crosscorrelation between $Y_t$ and $Z_t$:**
The crosscovariances:

$$\begin{aligned}\gamma_{12}(k) &= \text{Cov}(Y_{t+k}, Z_t) = \text{Cov}(1 + E_{t+k} - E_{t+k-1}, E_t) \\ &= \text{Cov}(E_{t+k}, E_t) - \text{Cov}(E_{t+k-1}, E_t) \\ &= \begin{cases} \sigma^2 & k = 0 \\ -\sigma^2 & k = 1 \\ 0 & \text{else} \end{cases}\end{aligned}$$

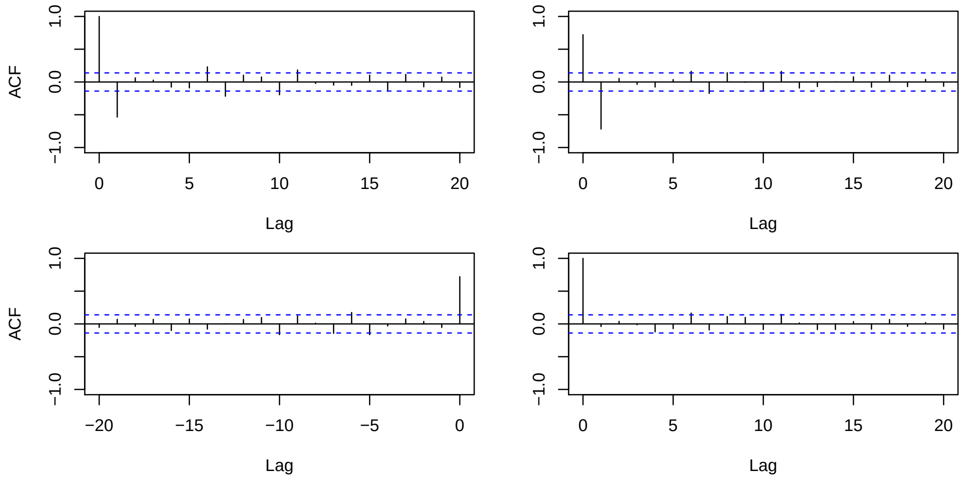
Thus, the crosscorrelations are given by

$$\rho_{12}(k) = \frac{\gamma_{12}(k)}{\sqrt{\gamma_{11}(0)\gamma_{22}(0)}} = \begin{cases} 1/\sqrt{2} = 0.71 & k = 0 \\ -1/\sqrt{2} = -0.71 & k = 1 \\ 0 & \text{else} \end{cases}$$

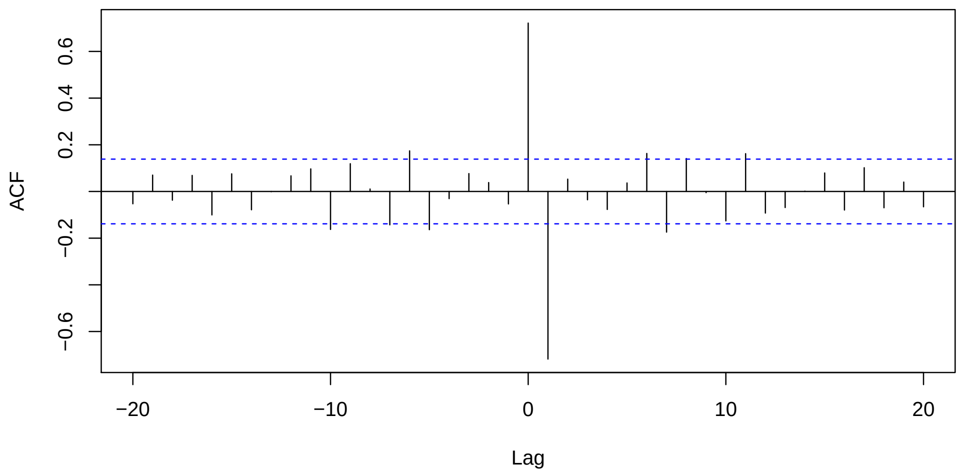
In this example the crosscorrelation $\rho_{12}(k)$ describes the relation between $Y_{t+k}$ (MA(1)-model) and $E_t$ (white noise). The crosscorrelation is always zero, except for lag 0 and lag 1.

**c) Simulation with R:**

```
> t.E <- ts(rnorm(201))
> t.X <- (1:201) + t.E
> t.Y <- diff(t.X)
> t.Z <- t.E
> acf(ts.intersect(t.Y,t.Z), ylim=c(-1,1))
```



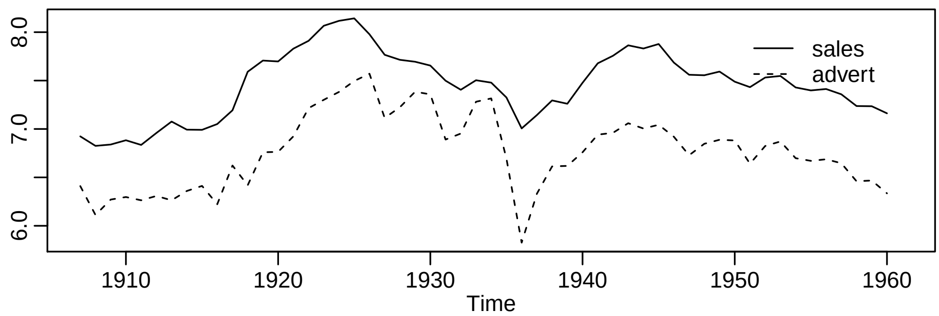

```
> ccf(t.Y,t.Z)
```




The simulated processes $Y_t$ and $Z_t$ behave as expected from theory.

**2.** **a)** The plots clearly show that the time series are *not* stationary:

```
> ts.plot(ts.sales, ts.advert, lty = 1:2)
> legend(c(1950,1950), c(7.1,8.1), legend = c("sales","advert"), lty=1:2, bty="n")
```

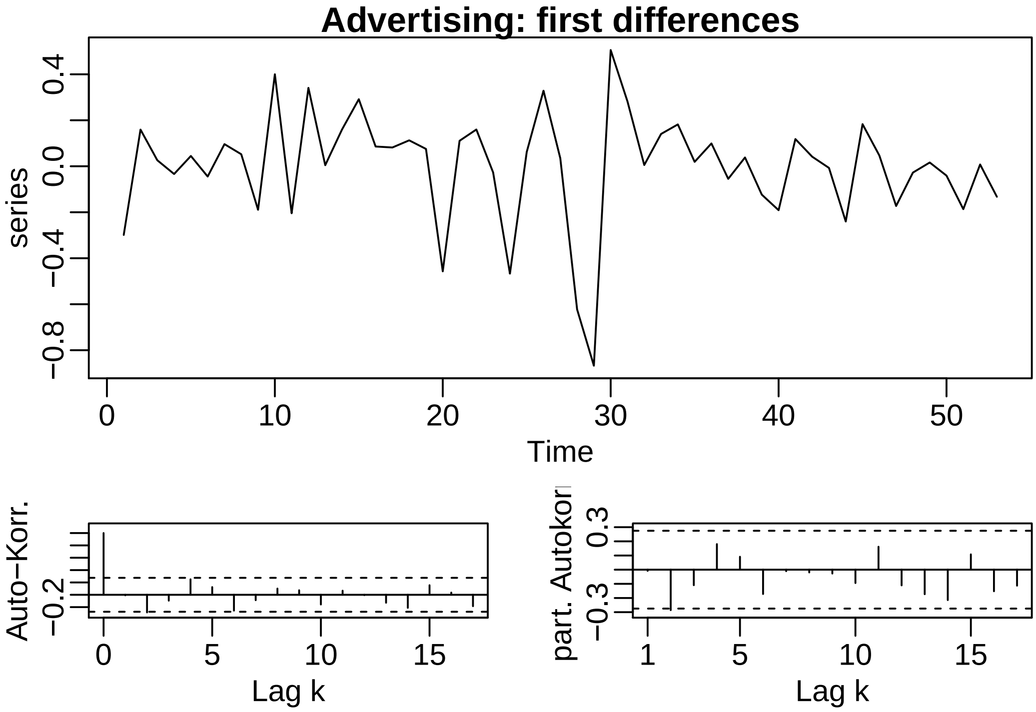



**b)** We first remove the missing values (last entry of the time series) and then calculate the first differences:

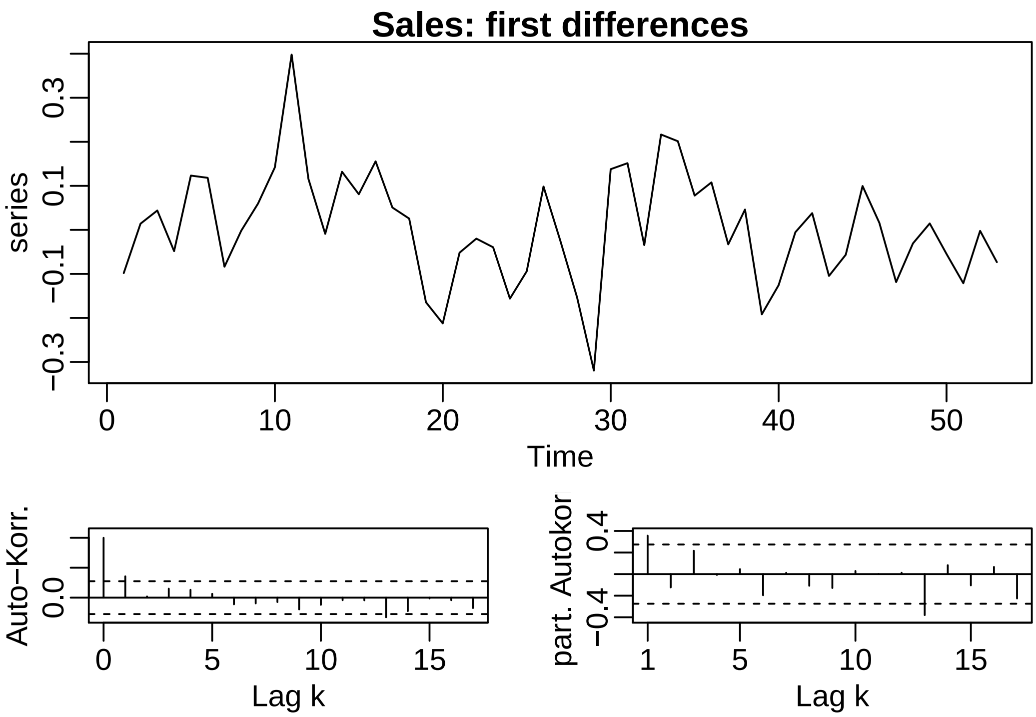
```
> ts.adv.d1 <- diff(ts.advert[!is.na(ts.advert)])
> ts.sal.d1 <- diff(ts.sales[!is.na(ts.sales)])
```

By differencing we can achieve stationarity as the following plots show (more or less):

```
> source("ftp://stat.ethz.ch/WBL/Source-WBL-2/R/f.acf.R")
> f.acf(ts.adv.d1, main="Advertising: first differences")
```




> *f.acf(ts.sal.d1, main="Sales: first differences")*




**c)** The transfer function model

$$Y_{2,t} = \sum_{j=0}^{\infty} \nu_j Y_{1,t-j} + E_t$$

makes the assumption that a change in the advertising expenditures $(Y_{1,t})$ causes a change in the (future) sales $(Y_{2,t})$, but *not* vice versa.

**d)** • From the correlogram of `d.adv.d1` we see that the input series $Y_{1,t} = X_{1,t} - X_{1,t-1}$ can be described as an $\mathrm{AR}(2)$ model. We fit it as follows:

```
> (r.fit.adv <- arima(ts.adv.d1, order = c(2, 0, 0)))
Call:
arima(x = ts.adv.d1, order = c(2, 0, 0))
```

```
Coefficients:
          ar1      ar2  intercept
      -0.0066  -0.2875    -0.0003
s.e.   0.1331   0.1314     0.0244

sigma^2 estimated as 0.05171:  log likelihood = 3.21,  aic = 1.59
```

Hence we get the model

$$Y_{1,t} = -0.0066 \cdot Y_{1,t-1} - 0.2875 \cdot Y_{1,t-2} + D_t \,,$$

where $D_t$ is a white noise with variance $\widehat{\sigma}_D^2 = 0.052$ (see component `r.fit.adv$sigma2`). The mean of the time series can be regarded as zero (one gets an estimate of $-0.0014$).
**Remark:** One could also fit the $\mathrm{AR}(2)$ model of the first differences with the function `ar.burg()` or `ar.yw()`, resp. The estimates of the coefficients are quite similar, though.

- We apply the transformation as in the lecture:

```
> ts.D <- resid(r.fit.adv)
> ts.Z <- filter(ts.sal.d1, c(1, -r.fit.adv$model$phi), sides = 1)
```

- In the transformed model

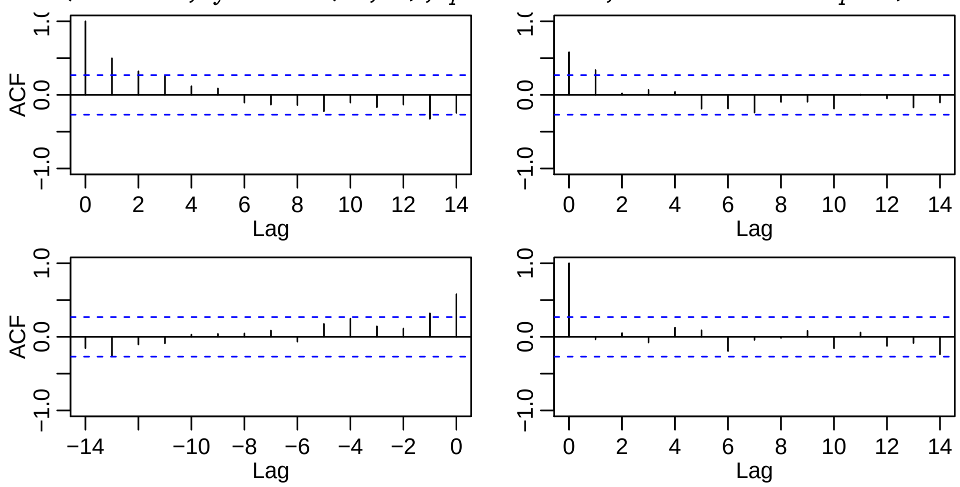
$$Z_t = \sum_{j=0}^{\infty} \nu_j D_{t-j} + U_t \ ,$$

the coefficients are the same as in the original transfer function model of part c). However, the time series $D_t$ is *uncorrelated* here. Hence we can estimate the coefficients $\nu_j$ by

$$\widehat{\nu}_k = \frac{\widehat{\gamma}_{21}(k)}{\widehat{\sigma}_D^2}, \quad k \geq 0$$

where $\widehat{\rho}_{21}(k)$ denotes the empirical cross correlations of $D_t$ and $Z_t$. The estimated coefficients $\widehat{\nu}_k$ are hence proportional to the empirical cross correlations $\widehat{\rho}_{21}(k)$ shown in the following plot.

```
> ts.trans <- ts.intersect(ts.Z, ts.D)
> acf(ts.trans, ylim = c(-1, 1), plot = TRUE, na.action = na.pass)
```




We see that $\widehat{\rho}_{21}(0)$ has the largest value. We find another large value at lag $k = -1$. This shows that, *contrary to our assumption* in part c), there is an influence of $Y_{2,t}$ on $Y_{1,t}$. Hence the modeling approach is not allowed since the prerequisites are not fulfilled. However, our analysis shows that there is a mutual influence between $Y_{2,t}$ and $Y_{1,t}$.
A change in the sales hence also causes a change in the advertising expenditures. This seems to be plausible in practice: the budget for advertising is usually established based on past sales, e.g. as a percentage of last year's sales.

- Estimation of the coefficients $\nu_j$ in R :

```
> gamma21 <- acf(ts.trans, plot = FALSE, type = "covariance",
   na.action = na.pass)$acf[, 1, 2]
> round(gamma21/r.fit.adv$sigma2, 2)[1:6]
[1]  0.33  0.20  0.01  0.04  0.02 -0.11
```