4) Computing PC's

$\sqrt{5}$ (1)

$R = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$; in example: $r = 0{,}71$

Eigendecomp: Find λ's & v's with $Rv = \lambda v$ of R

1.1) Compute Eigenvalues (= diagonal entries of R in a coordinate system where R is diag.)

$$\det(R - \lambda \mathbb{1}) \overset{!}{=} 0$$

$$\det\left(\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \det \begin{pmatrix} 1-\lambda & r \\ r & 1-\lambda \end{pmatrix} =$$

$$= (1-\lambda)^2 - r^2 = 1 - 2\lambda + \lambda^2 - r^2 = \lambda^2 - 2\lambda + 1 - r^2$$

$$\Rightarrow \lambda_{1/2} = \frac{2 \pm \sqrt{4 - 4 \cdot (1-r^2)}}{2} = \frac{2 \pm \sqrt{4r^2}}{2} = 1 \pm r$$

in example: $\lambda_{1/2} = 1 \pm 0{,}71 \Rightarrow \lambda_1 = 1{,}71;\ \lambda_2 = 0{,}29$ $\Rightarrow$ Std. dev along PC1 $\sqrt{1{,}71} = 1{,}307$ PC2 $\sqrt{0{,}29} = 0{,}538$

$\Rightarrow$ There is some basis, in which R looks like this: $\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$

1.2) Compute Eigenvectors

$$Rv \overset{!}{=} \lambda_i v$$

a) $\lambda_1 = 1 + r$

$$\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1+r) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v_1 + r v_2 \\ r v_1 + v_2 \end{pmatrix} = \begin{pmatrix} v_1 + r \cdot v_1 \\ v_2 + r v_2 \end{pmatrix} \Rightarrow v_2 = v_1$$

Normalization: $v^T v = 1 \Rightarrow (v_1\ \ v_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^2 + v_2^2 = 1$

Using $v_1 = v_2 \Rightarrow 2 v_1^2 = 1 \Leftrightarrow v_1 = 1/\sqrt{2} = v_2$

$$\Rightarrow v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0{,}71 \\ 0{,}71 \end{pmatrix} = 1.\ PC \quad ; R v_1 = \lambda_1 \cdot v_1$$

b) $\lambda_2 = 1 - r$ same calculation: $v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -0{,}71 \\ 0{,}71 \end{pmatrix} = 2.\ PC$

$$R v_2 = \lambda_2 v_2$$

2) Change of basis

$$\begin{tikzcd}
\text{Std} & \bullet \arrow[r, "R"] \arrow[d, "A^T"'] & \bullet \\
\text{PC} & \bullet \arrow[r, "\Lambda"'] & \bullet \arrow[u, "A"']
\end{tikzcd}$$

A has PC's as columns

$$A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$\uparrow$ 1. PC $\qquad \uparrow$ 2. PC

$A^T A = \mathbb{1}$

$\Downarrow$

A is orthogonal,

Loadings

Represent R in basis of PC's:

$$\Lambda = A^T R A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\cdot\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1+r & r-1 \\ r+1 & 1-r \end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} 1+r+r+1 & r-1+1-r \\ -1-r+r+1 & -r+1+1-r \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2+2r & 0 \\ 0 & 2-2r \end{pmatrix} = \begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$$

Variance along 1. PC $\qquad$ Variance along 2. PC

trace (R) = sum of diagonal elements

Equal for all bases $\Rightarrow$ trace (R) = total variance

3) Express new point in PC basis

$x_{PC} = A^T x$; In example: $\mu_{x_1} = 185{,}7$, $\mu_{x_2} = 183{,}8$, $\sigma_{x_1} = 9{,}76$, $\sigma_{x_2} = 10{,}04$

$$x = \begin{pmatrix} 191 \\ 179 \end{pmatrix} \xrightarrow{\text{Scale}} x_s = \begin{pmatrix} \frac{191-\mu x}{\sigma x} \\ \frac{179-\mu x}{\sigma_y} \end{pmatrix} = \begin{pmatrix} 0{,}54 \\ -0{,}48 \end{pmatrix}$$

$$x_{PC} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0{,}54 \\ -0{,}48 \end{pmatrix} = \begin{pmatrix} 0{,}04 \\ -0{,}72 \end{pmatrix}$$

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