Applied Time Series Analysis
FS 2012 – Week 06

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ETH Zürich, March 26, 2012

Looking Back & Outlook

We did consider **AR(p)-models**

$$(X_t - \mu) = \alpha_1 (X_{t-1} - \mu) + ... + \alpha_p (X_{t-p} - \mu) + E_t$$

where the correlation structure was as follows:

ACF: **„exponential decay“**
PACF: **= 0 for all lags k>p**

Now, in practice we could well observe a time series whose autocorrelation differs from the above.

We will thus discuss ARMA-models, a class that is suitable for modeling a wider spectrum of dependency structures.

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Backshift Operator

Purpose: more convenient notation

What it is: function $B(\cdot)$
„go back 1 observation“
„increment the time series index t by -1“

Examples: $B(X_t) = X_{t-1}$, or also $BX_t = X_{t-1}$

and

$$B^3 X_t = B(B(B(X_t))) = B(B(X_{t-1}) = B(X_{t-2}) = X_{t-3}$$

Using the Backshift Operator

The backshift operator allows for convenient differencing:

a) First order difference with lag 1

$$Y_t = X_t - X_{t-1} = (1-B)X_t$$

b) Second order difference with lag 1

$$Z_t = Y_t - Y_{t-1} = (1-B)Y_t = (1-B)^2 X_t = (1-2B+B^2)X_t$$

c) b[th] order difference with lag a

$$W_t = (1-B^a)^b X_t$$

Moving Average Models

Whereas for AR(p) models, the current observation of a time series is written as a linear combination of its own past, MA(q) models can be seen as an extension of the „pure" model

$$X_t = E_t \text{ , where } E_t \text{ is a white noise process,}$$

in the sense that past innovation terms $E_{t-1}, E_{t-2}, \ldots$ are included, too. We call this a moving average model:

$$X_t = E_t + \beta_1 E_{t-1} + \beta_2 E_{t-2} + \ldots + \beta_q E_{t-q}$$

Note that there are other interpretations, too. We will discuss them later.

Notation for MA(q)-models

The backshift operator, and the characteristic polynom, allow for convenient notation:

MA(q): $X_t = E_t + \beta_1 E_{t-1} + \beta_2 E_{t-2} + ... + \beta_q E_{t-q}$

MA(q) with BS: $X_t = \left(1 + \beta_1 B + \beta_2 B^2 + ... + \beta_q B^q\right) E_t$

MA(q) with BS+CP: $X_t = \Theta(B) E_t$

where

$$\Theta(z) = 1 + \beta_1 z + \beta_2 z^2 + ... + \beta_q z^q$$

is the characteristic polynom

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Stationarity of MA(1)-Models

We first restrict ourselves to the simple MA(1)-model

$$X_t = E_t + \beta_1 E_{t-1}, \text{ where } E_t \text{ is an innovation}$$

The series X_t is weakly stationary, no matter what the choice of the parameter β_1 is.

Remember that for proving this, we have to show that:

- the expected value is 0
- the variance is constant and finite
- the autocovariance only depends on the lag k

→ **see the blackboard for the proof**

ACF of the MA(1)-Process

We can deduct the ACF for the MA(1)-process:

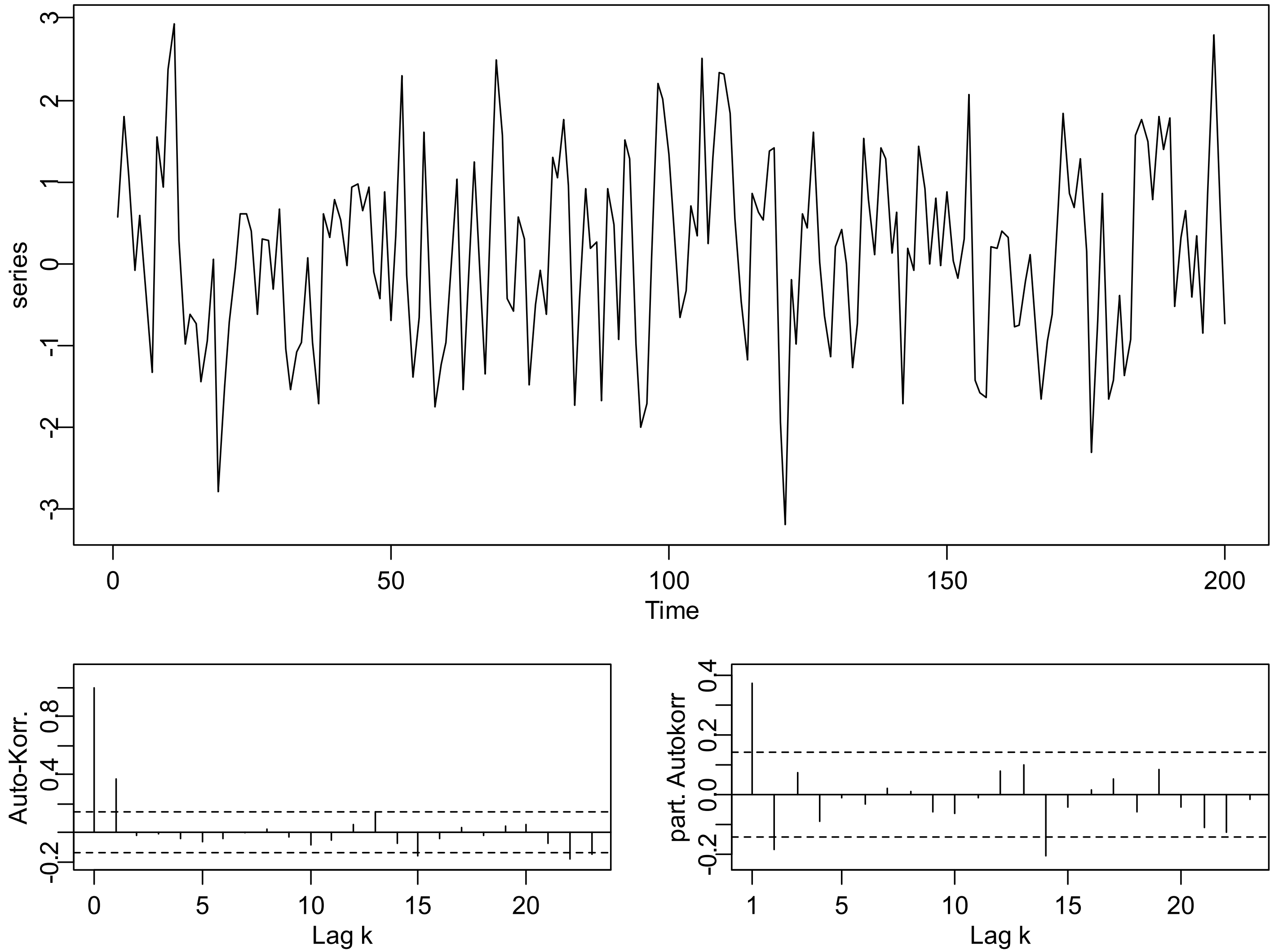
$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\beta_1}{(1+\beta_1^2)}$$

and

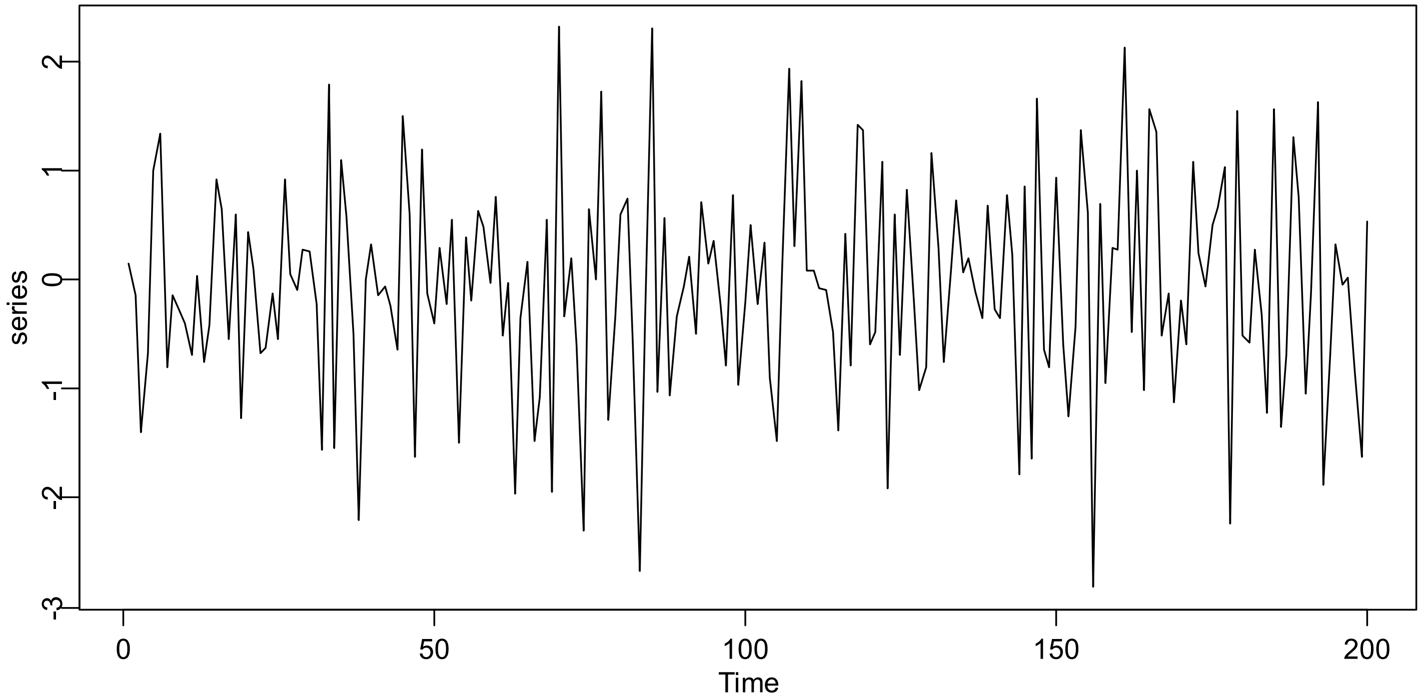
$$\rho(k) = 0 \quad \text{for all k>1.}$$

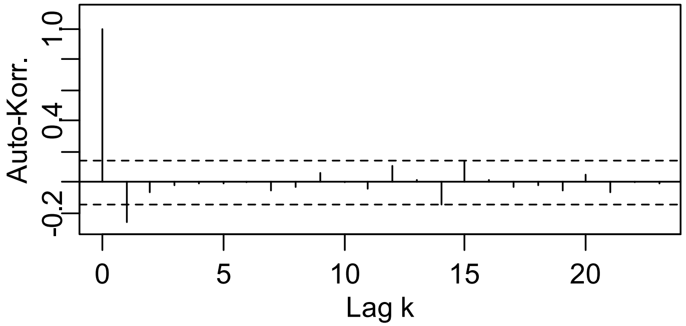
Thus, similar behavior to the PACF of an AR(1).

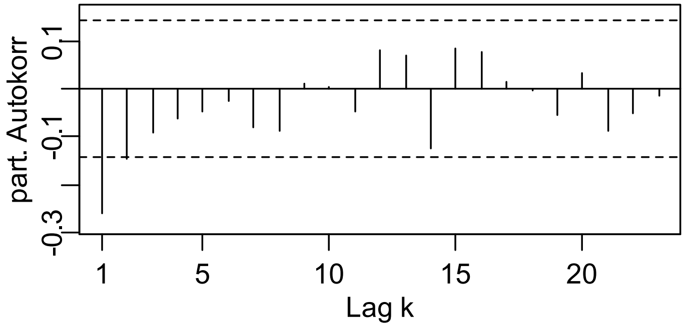
Simulated Process with β_1=0.5



Simulated Process with β_1=-0.5







MA(1): Remarks

Without additional assumptions, the ACF of an MA(1) doesn‘t allow identification of the generating model.

In particular, the two processes

$$X_t = E_t + 0.5 \cdot E_{t-1}$$

$$U_t = E_t + 2 \cdot E_{t-1}$$

have identical ACF:

$$\rho(1) = \frac{\beta_1}{1+\beta_1^2} = \frac{1/\beta_1}{1+(1/\beta_1^2)}$$

MA(1): Invertibilty

- An MA(1)-, or in general an MA(q)-process is said to be invertible if the roots of the characteristic polynom lie outside of the unit circle.
- Under this condition, there exists only one MA(q)-process for any given ACF.
- This translates to restrictions on the coefficients. For a MA(1)-model, $|\beta_1| < 1$ is required.
- **See blackboard for further explanation...**

Invertible MA(1) = AR(∞)

Invertible MA(1)-processes can be written as an AR(∞):

$$X_t = E_t + \beta_1 E_{t-1}$$

$$= E_t + \beta_1 (X_{t-1} - \beta_1 E_{t-2})$$

$$= E_t + \sum_{j=1}^{\infty} \psi_j X_{t-j}$$

MA(1): More Remarks

- MA(1)-processes have mean zero: $E[X_t] = 0$
- If an observed time series shows MA(1)-properties in ACF/PACF, but has a mean different from zero, we can always model the centered series (idem AR(p)).
- For an MA(1)-process,

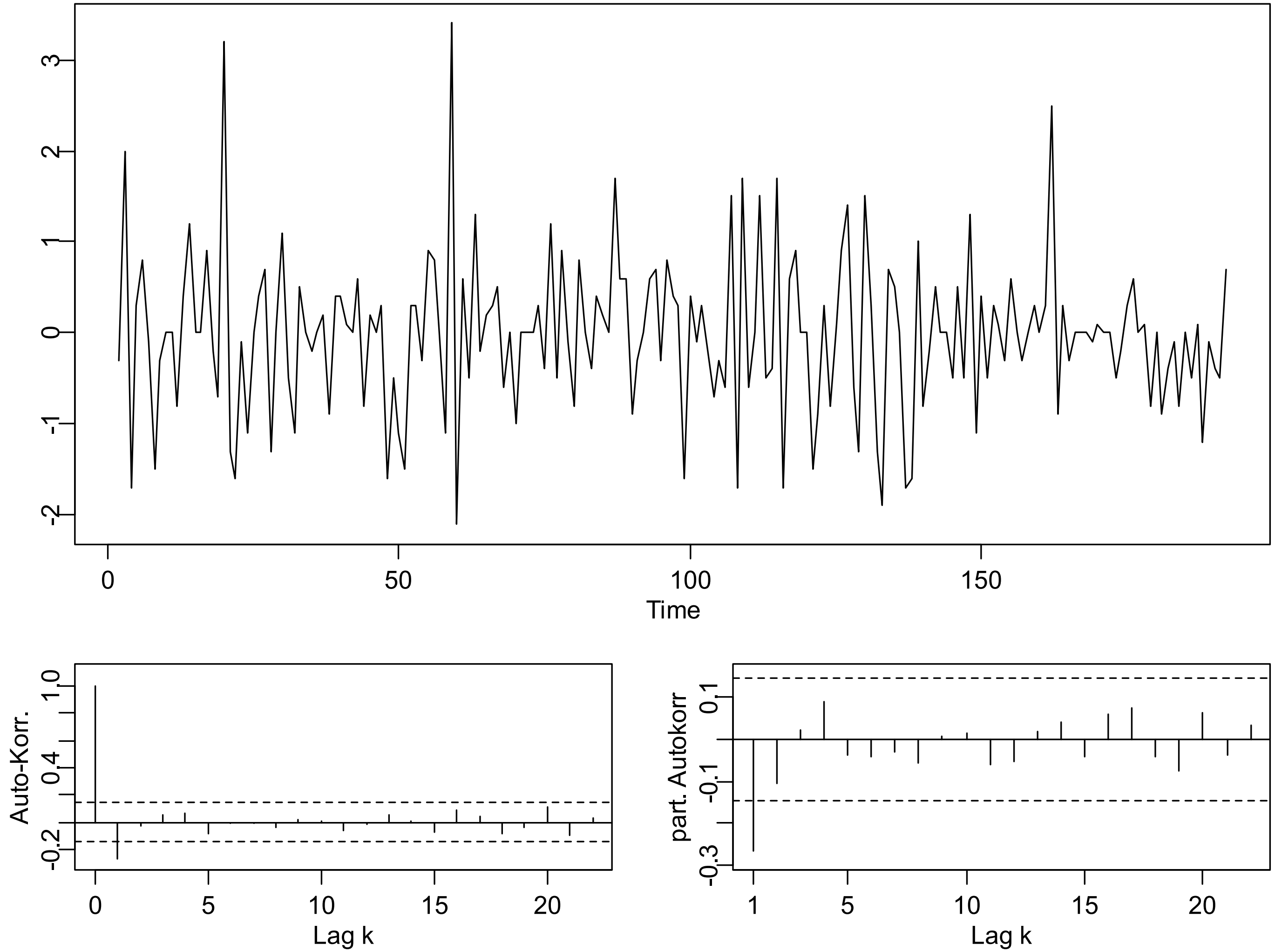
  $$|\rho(1)| \leq 0.5$$

  always holds. If the estimated first ACF-coefficient clearly exceeds 0.5, this is counter-evidence to a MA(1).

MA(1): Example

- daily return of an AT&T bond from 04/1975 to 12/1975
- the time series has 192 observations
- we are looking at the first-order differences
- an MA(1) model seems to fit the data (→ next slide)
- since we are looking at a differenced series, this is in fact an ARIMA(0,1,1) model (→ will be discussed later…)

MA(1): Example


3
2
1
0
-1
-2
0
50
100
150
Time
Auto-Korr.
1.0
0.4
-0.2
0
5
10
15
20
Lag k
part. Autokorr
0.1
-0.1
-0.3
1
5
10
15
20
Lag k

MA(q)-Models

The MA(q)-model is defined as follows:

$$X_t = E_t + \beta_1 E_{t-1} + \beta_2 E_{t-2} + \ldots + \beta_q E_{t-q} \ ,$$

where E_t are i.i.d. innovations (=a white noise process).

The ACF of this process can be computed from the coefficients:

$$\rho(k) = \frac{\sum_{i=0}^{q-k} \beta_i \beta_{i+k}}{\sum_{i=0}^{q} \beta_i^2} \ , \quad \text{for all k=1,…, q with } \beta_0 = 1$$

$$\rho(k) = 0 \ , \qquad \text{for all k>q}$$

ACF/PACF of MA(q)

ACF

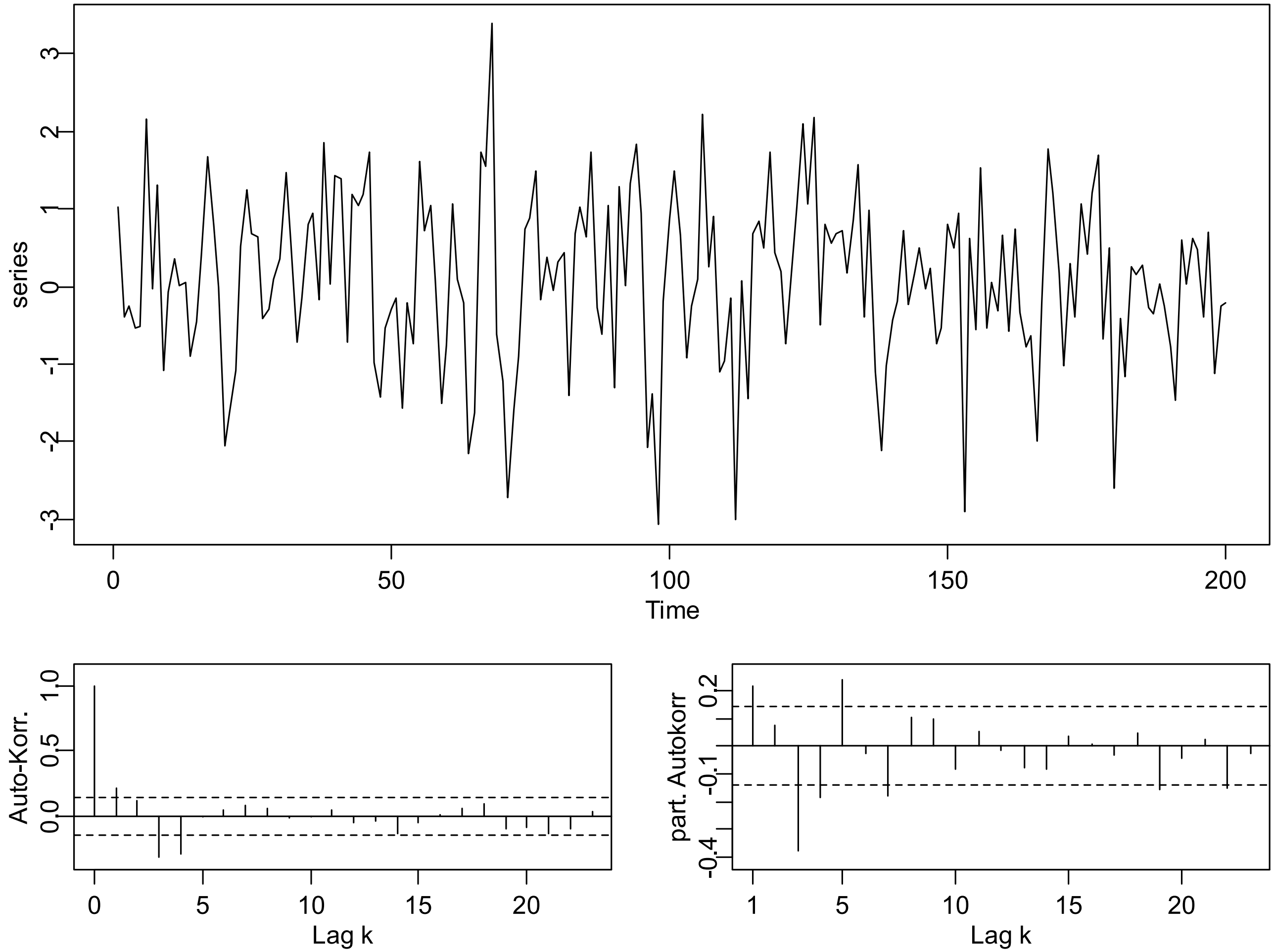
- the ACF of an MA(q) has a cut-off at lag k=q
- it behaves thus like the PACF of an AR(q)-model

PACF

- the PACF is (again) complicated to determine, but:
- the PACF of an MA(q) has an „exponential decay"
- it behaves thus like the ACF of an AR-model

MA(4): Example

$$X_t = E_t + 0.3 \cdot E_{t-1} + 0.3 \cdot E_{t-2} - 0.2 \cdot E_{t-3} - 0.2 \cdot E_{t-4}, \quad E_t \sim N(0,1)$$



ARMA(p,q)-Models

An ARMA(p,q)-model combines AR(p) and MA(q):

$$X_t = \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + E_t + \beta_1 E_{t-1} + \ldots + \beta_q E_{t-q}$$

where E_t are i.i.d. innovations (=a white noise process).

It‘s easier to write an ARMA(p,q) with the characteristic polynom:

$\Phi(B) X_t = \Theta(B) E_t$, where

$\Phi(z) = 1 - \alpha_1 z - \ldots \alpha_p z^p$ is the cP of the AR-part, and

$\Theta(z) = 1 - \beta_1 z - \ldots \beta_q z^q$ is the cP of the MA-part

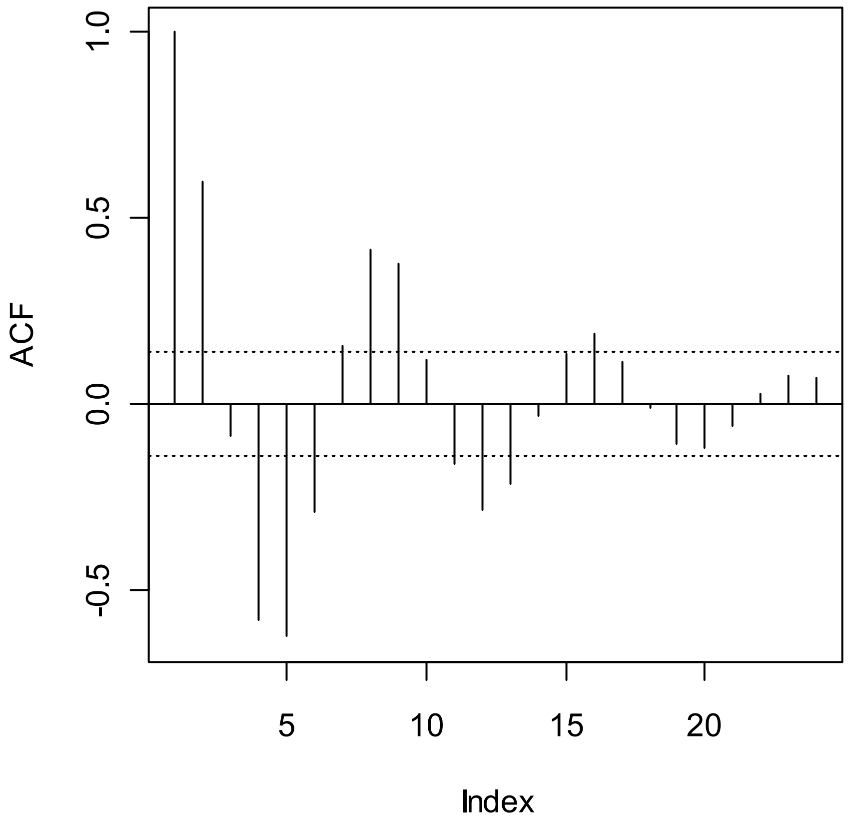
Stationarity/Invertibility of ARMA(p,q)

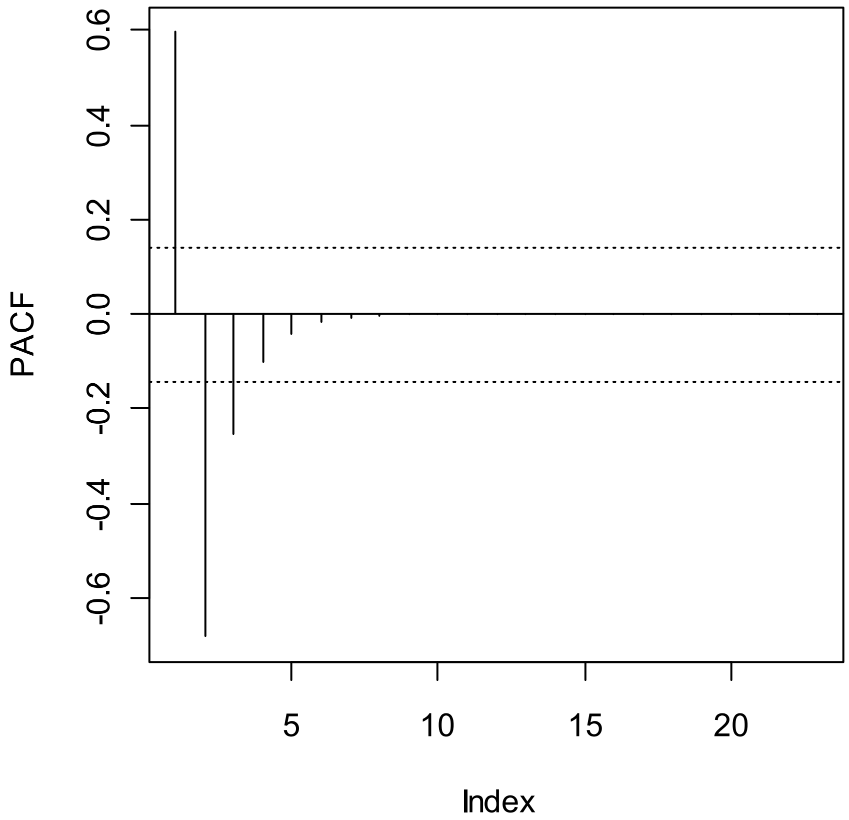
- both properties are determined by the cP
- the AR-cP determines stationarity
- the MA-cP determines invertibility
- condition: roots of the cP outside of the unit circle

- stationarity: model can be written as a MA(∞)
- invertibility: model can be written as an AR(∞)

True ACF/PACF of an ARMA(2,1)

$$X_t = 1.2 \cdot X_{t-1} - 0.8 \cdot X_{t-2} + E_t + 0.4 \cdot E_{t-1}, \;\; E_t \sim N(0,1)$$





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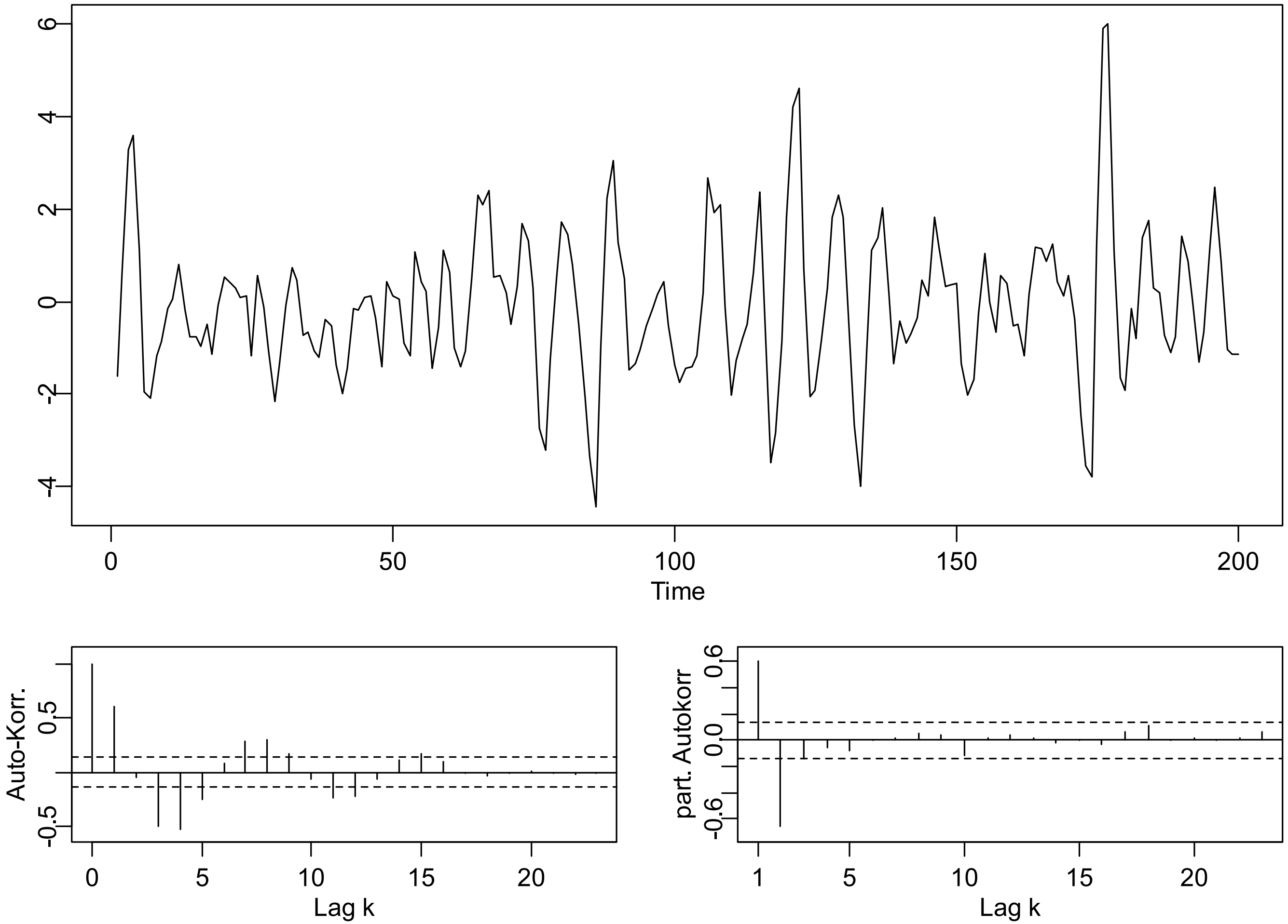
Simulated ACF/PACF of an ARMA(2,1)

$$X_t = 1.2 \cdot X_{t-1} - 0.8 \cdot X_{t-2} + E_t + 0.4 \cdot E_{t-1},\ E_t \sim N(0,1)$$



Properties of ACF/PACF in ARMA(p,q)

	ACF	PACF
AR(p)	exponential decay	cut-off at lag p
MA(q)	cut-off at lag q	exponential decay
ARMA(p,q)	as AR(p) for k>q	as MA(q) for k>p

→ all linear time series processes can be approximated by an ARMA(p,q) with possibly large p,q. They are thus are very rich class of models.

Fitting ARMA(p,q)

What needs to be done?

1) **Achieve stationarity**
→ transformations, differencing, modeling, …

2) **Choice of the order**
→ determining (p,q)

3) **Parameter estimation**
→ Estimation of $\alpha, \beta, \mu, \sigma_E^2$

4) **Residual analysis**
→ if necessary, repeat 1), and/or 2)-4)

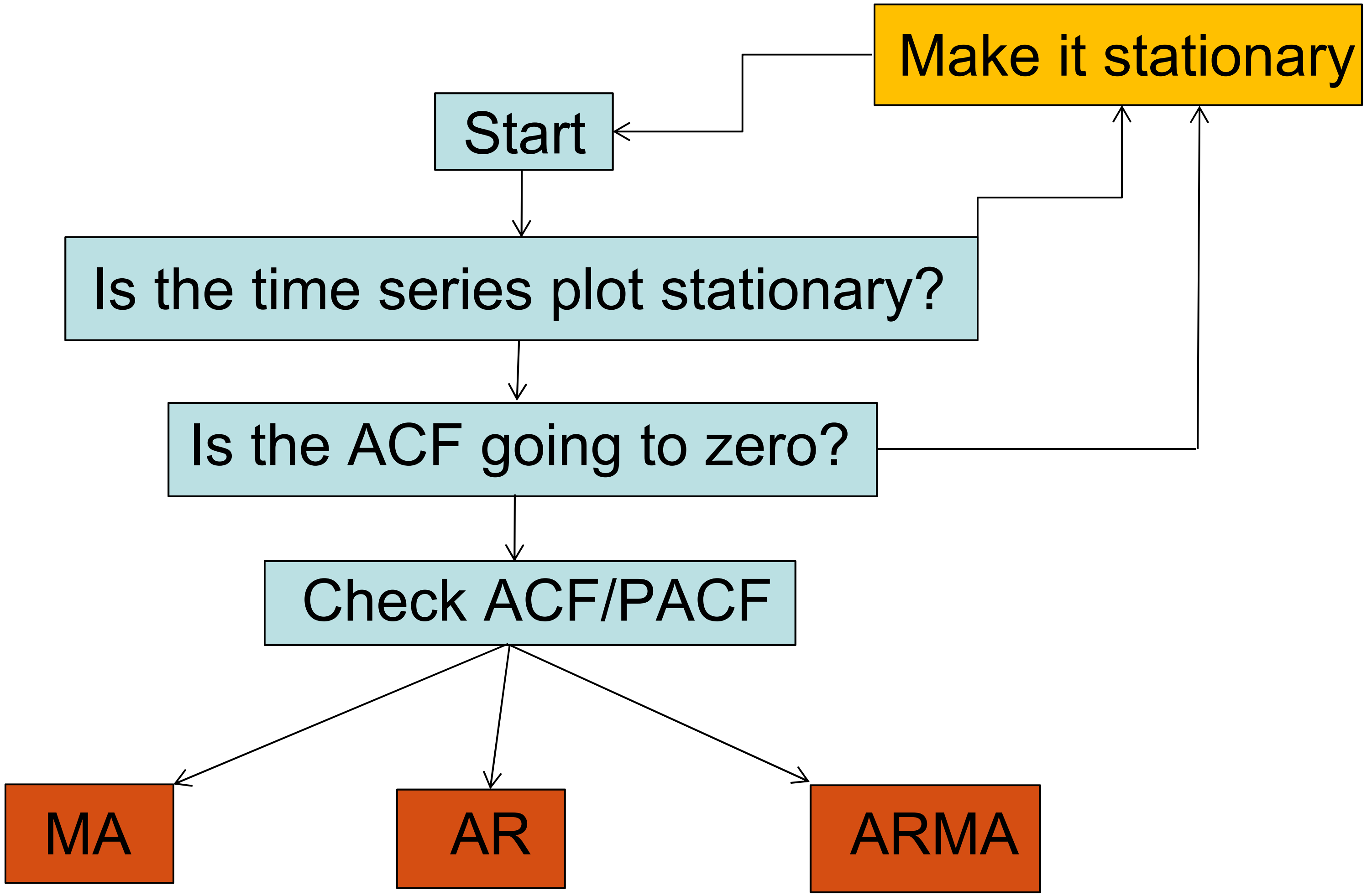
Identification of the Order (p,q)

Please note:

- We only have one single realization of the time series with finite length.
- The plots (etc.) we base the order choice on are not „facts“, but are estimations with uncertainty.
- This holds especially for the ACF/PACF plots.
- Every ARMA(p,q) can be written as AR(∞) or MA(∞)

→ **There is usually >1 model that describes the data well**.

ARMA(p,q)-Modeling



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Parameter Estimation

For parameter estimation with AR(p) models, we had 4 choices:

a) Regression
b) Yule-Walker
c) Maximum-Likelihood
d) Burg‘s Algorithm

For ARMA(p,q) models, only two options are remaining, and both of them require numerical optimization:

1) Conditional Sum of Squares
2) Maximum-Likelihood

Conditional Sum of Squares

Idea: This is an iterative approach where the parameters are determined such that the sum of squared errors (between observations and fitted values) are minimal.

$$S(\hat{\beta}_1, ..., \hat{\beta}_q) = \sum_{t=1}^{n} \hat{E}_t^2 = \sum_{t=1}^{n} (X_t - (\hat{\beta}_1 \hat{E}_{t-1} - ... - \hat{\beta}_1 \hat{E}_{t-q})^2$$

This requires starting values which are chosen as:

$$\hat{E}_0 = 0,\ \hat{E}_{-1} = 0,\ ...,\ \hat{E}_{1-q} = 0$$

A numerical search is used to find the parameter values that minimize the entire conditional sum of squares. They also serve as starting values for MLE.

Maximum-Likelihood-Estimation

Idea: Determine the parameters such that, given the observed time series $x_1,\ldots,x_n$, the resulting model is the most plausible (i.e. the most likely) one.

→ **This requires the choice of a probability distribution for the time series $X = (X_1, \ldots, X_n)$**

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Maximum-Likelihood-Estimation

If we assume the ARMA(p,q)-model

$$X_t = \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + E_t + \beta_1 E_{t-1} + \ldots + \beta_q E_{t-q}$$

and i.i.d. normally distributed innovations

$$E_t \sim N(0, \sigma_E^2)$$

the time series vector has a multivariate normal distribution

$$X = (X_1, \ldots, X_n) \sim N(\mu \cdot \underline{1}, V)$$

with covariance matrix V that depends on the model parameters α, β and σ_E^2.

Maximum-Likelihood-Estimation

We then maximize the density of the multivariate normal distribution with respect to the parameters

$$\alpha, \beta, \ \mu \text{ and } \sigma_E^2.$$

The observed x-values are hereby regarded as fixed values.

→ **This is a highly complex non-linear optimization problem that requires sophisticated algorithms and starting values which are usually provided by CSS (at least that's the default in R's `arima()`).**

Maximum-Likelihood-Estimation

```
> r.Pmle <- arima(d.Psqrt,order=c(2,0,0),include.mean=T)
> r.Pmle

Call: arima(x=d.Psqrt, order=c(2,0,0), include.mean=T)

Coefficients:

        ar1    ar2  intercept
      0.275  0.395      3.554
s.e.  0.107  0.109      0.267

sigma^2 = 0.6:  log likelihood = -82.9,  aic = 173.8
```

MLE: Remarks

- The MLE approach would work for any distribution. However, for innovation distributions other than Gaussian, the joint distribution might be „difficult“.

- For „reasonable“ deviations from the normality assumption, MLE still yields „good“ results.

- Besides the parameter estimates, we also obtain an estimate of their standard error

- Other software packages such as for example SAS don't rely on MLE, but use CSS, which is in spirit similar to Burg's algorithm.

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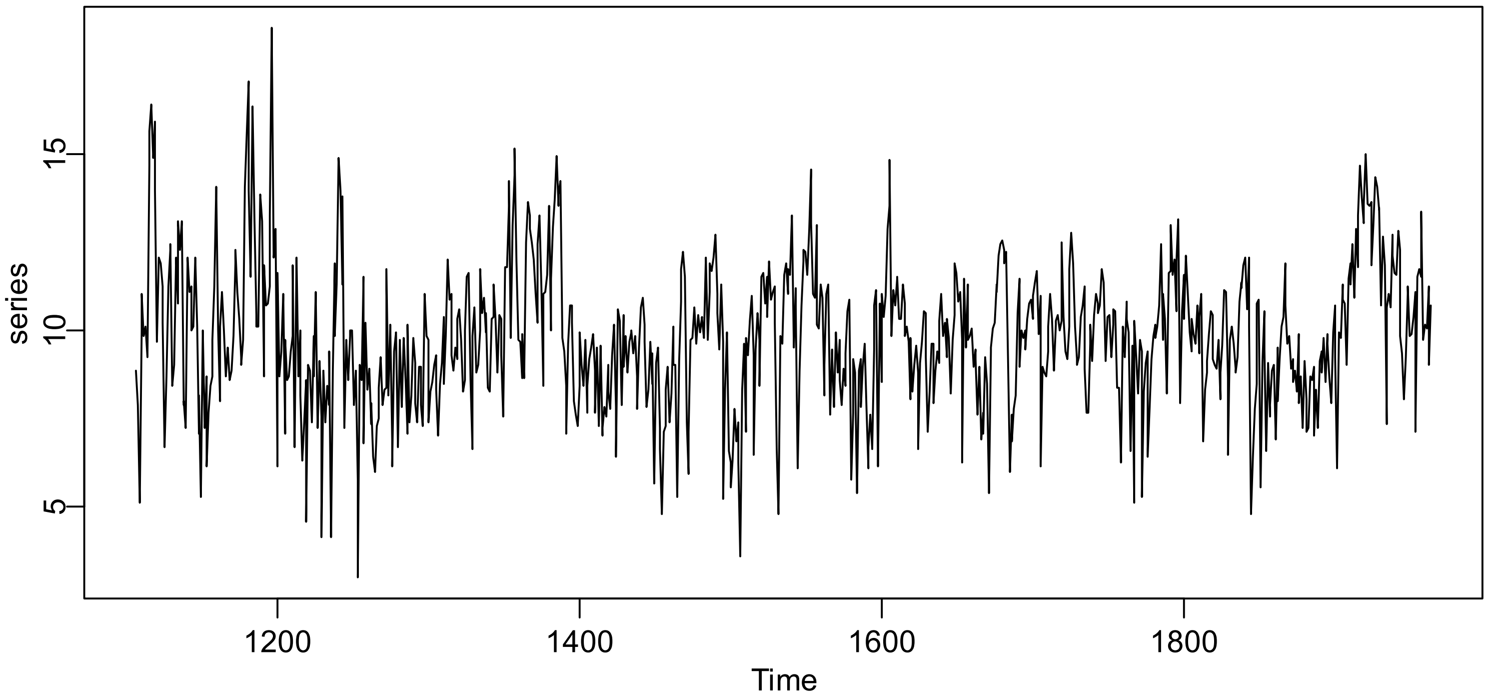
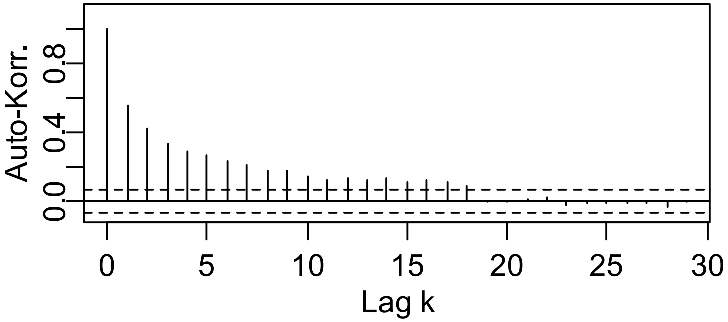
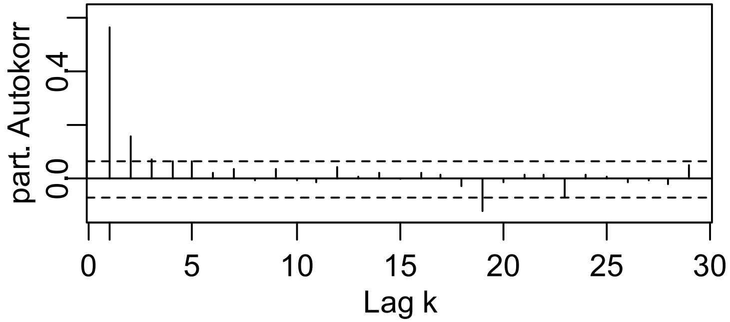
Douglas Fir: Original Data







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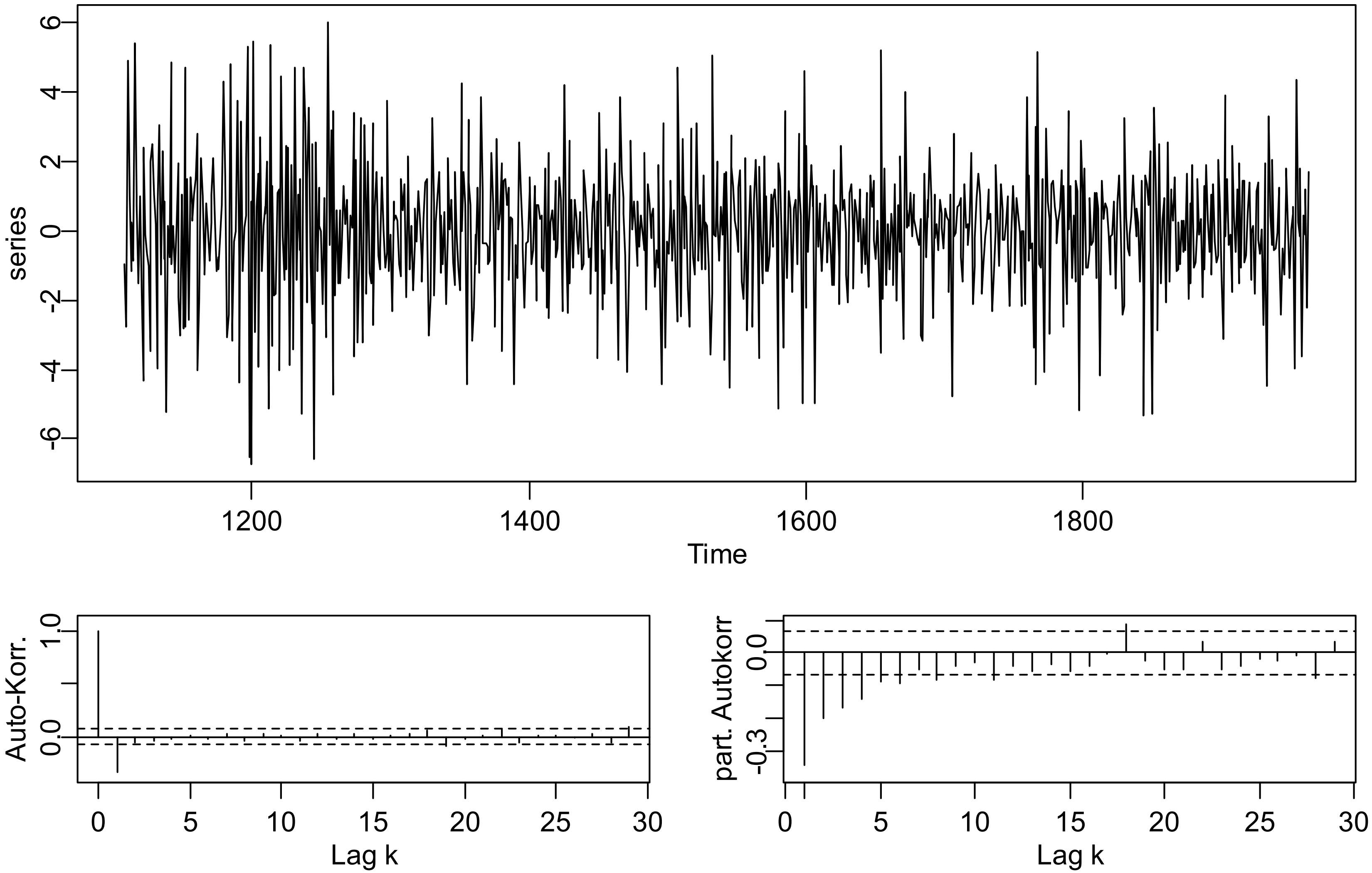
Douglas Fir: Differenced Series



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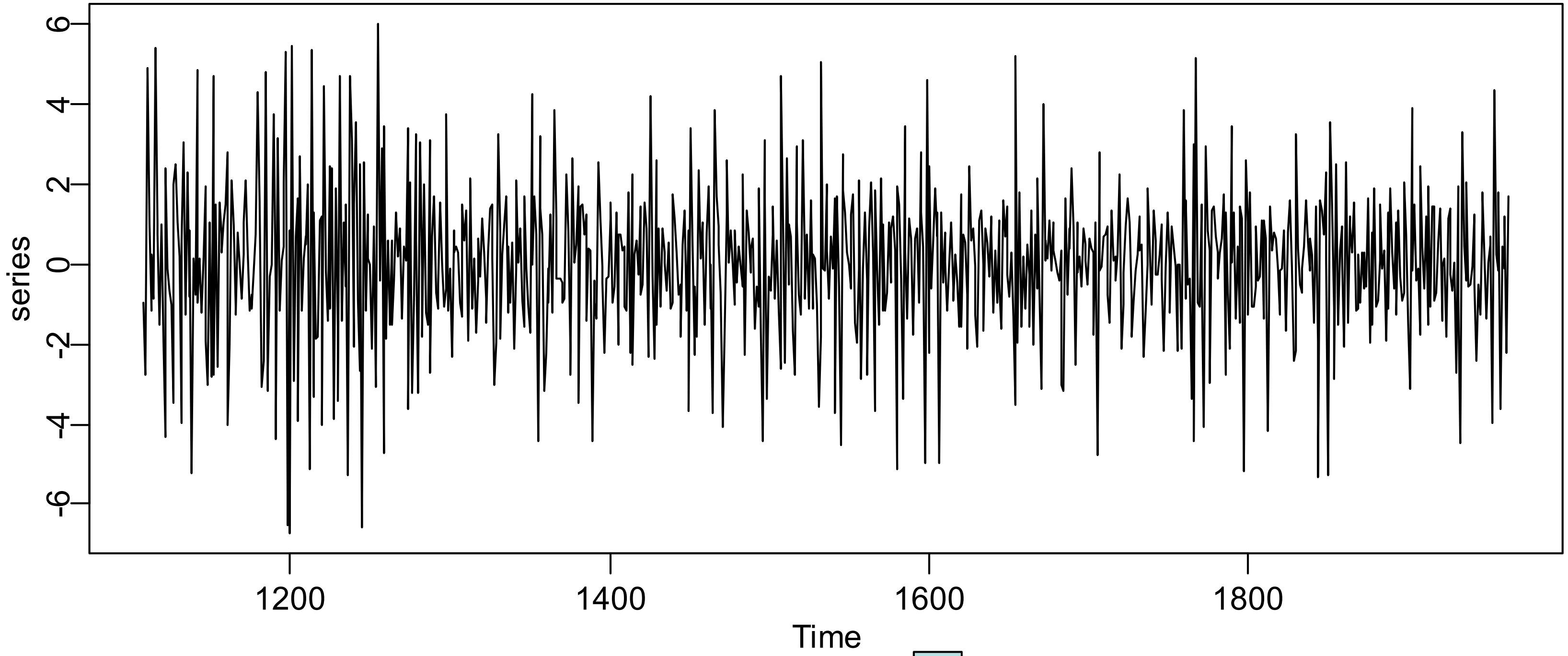
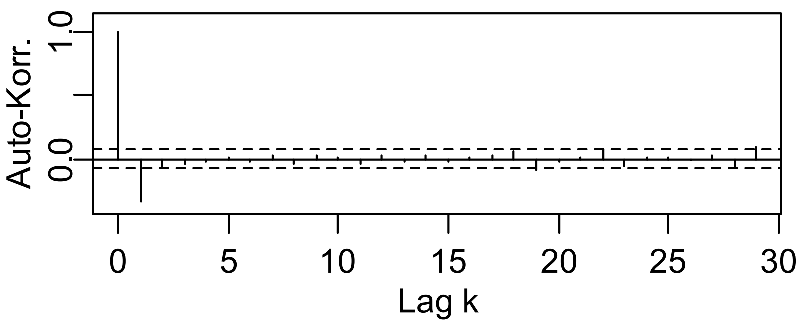
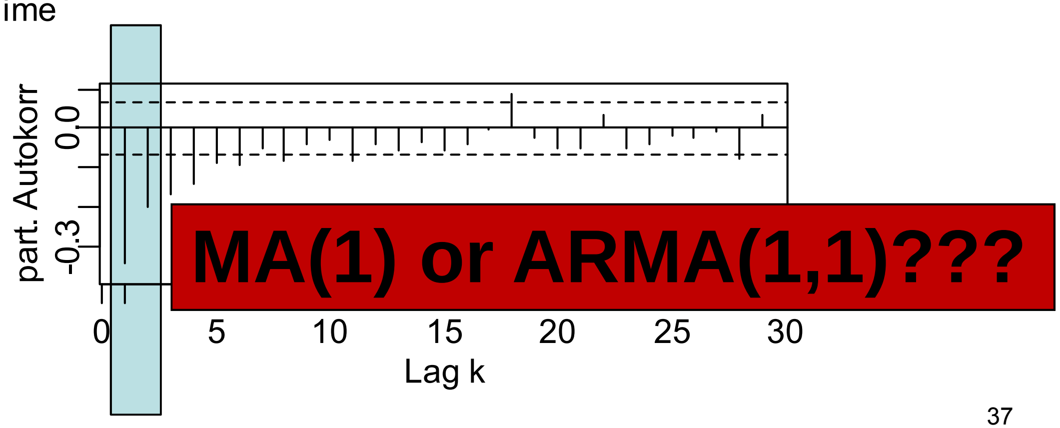
Douglas Fir: Differenced Series







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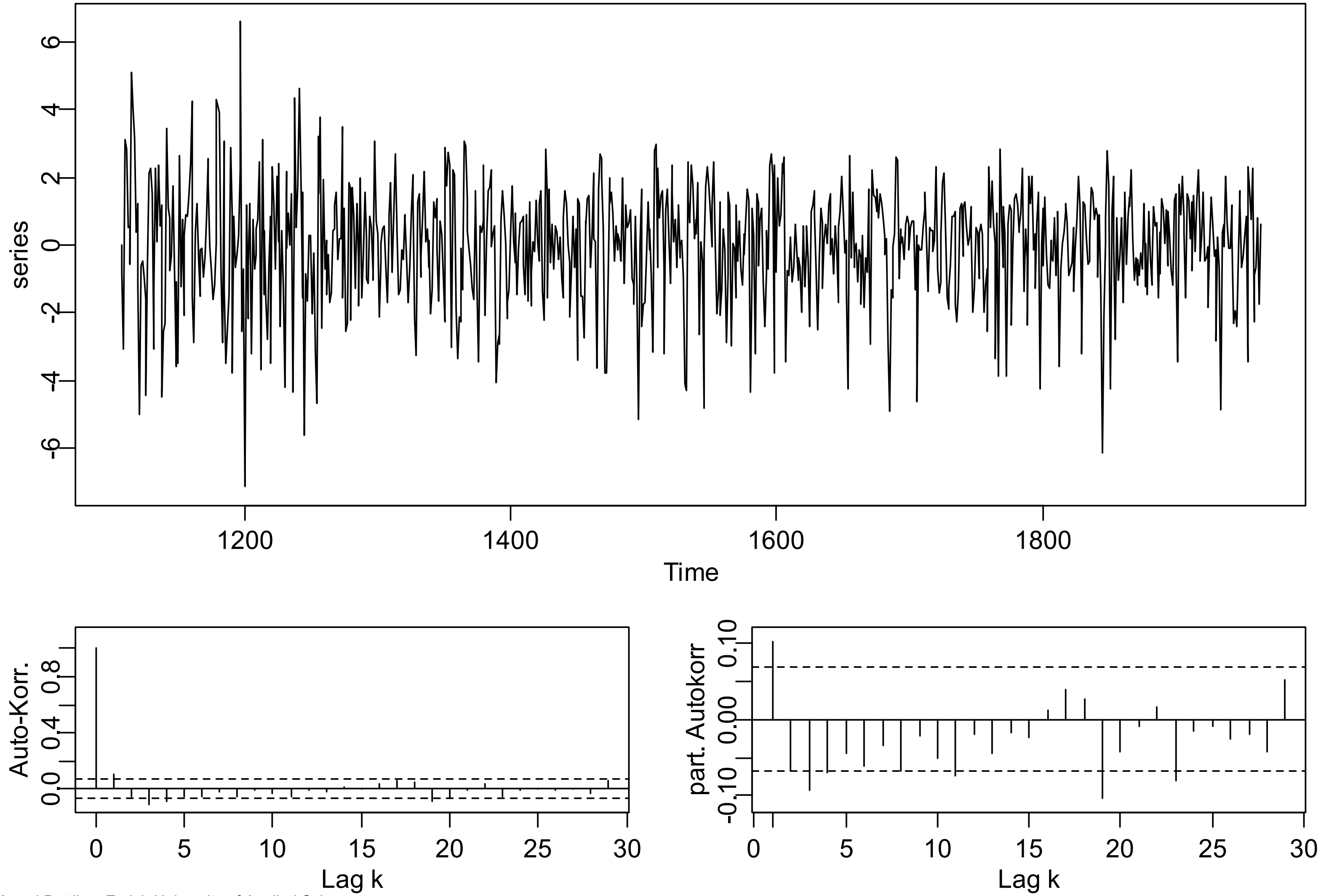
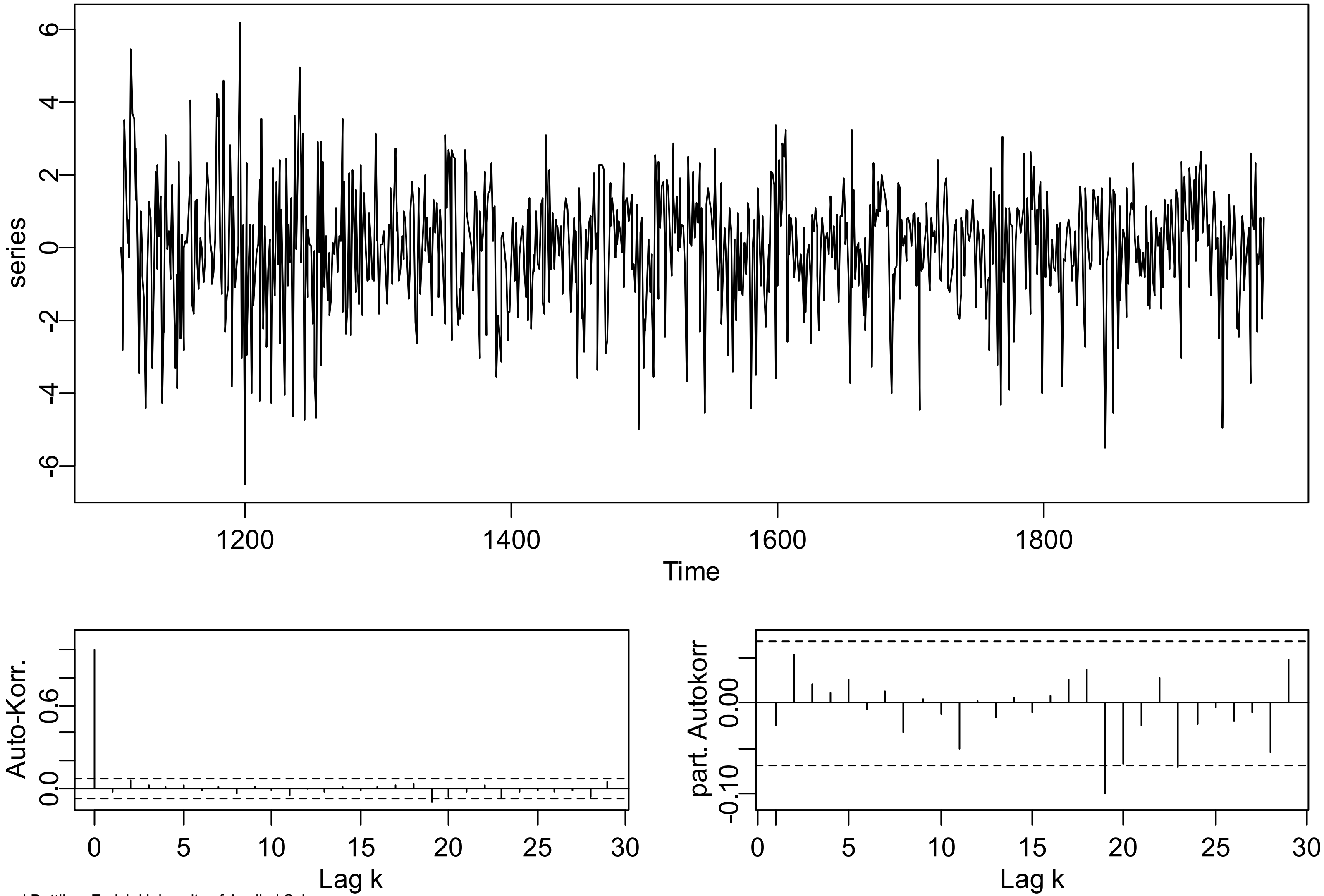
Residuals of MA(1)



Residuals of ARMA(1,1)



Another Example: Fitting ARMA(p,q)

What needs to be done?

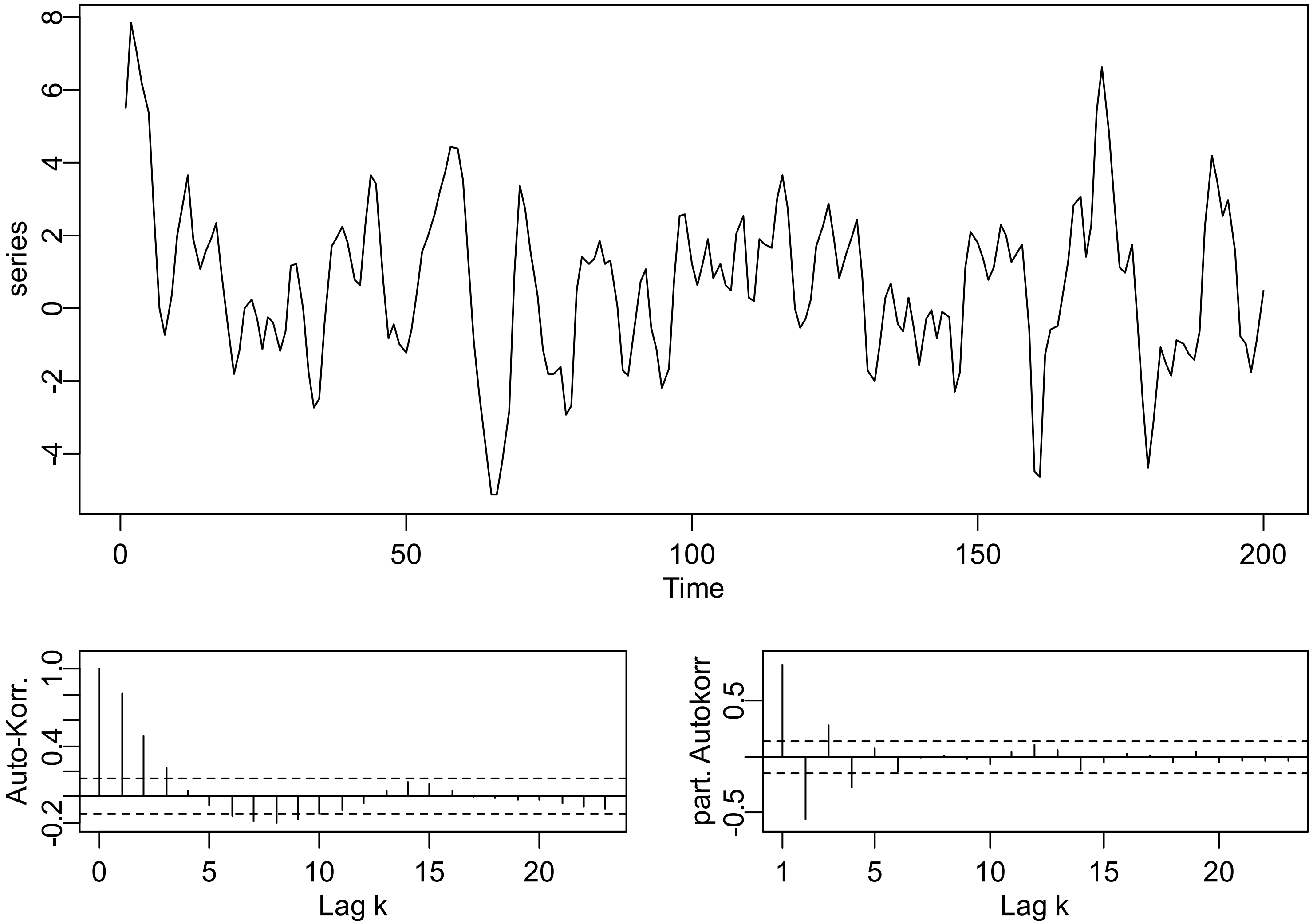
1) Achieve stationarity
→ transformations, differencing, modeling, …

2) Choice of the order
→ determining (p,q), plus integration order d for ARIMA

3) Parameter estimation
→ ML-estimation of $\alpha, \beta, \mu, \sigma_E^2$

4) Residual analysis
→ if necessary, repeat 1), and/or 2)-4)

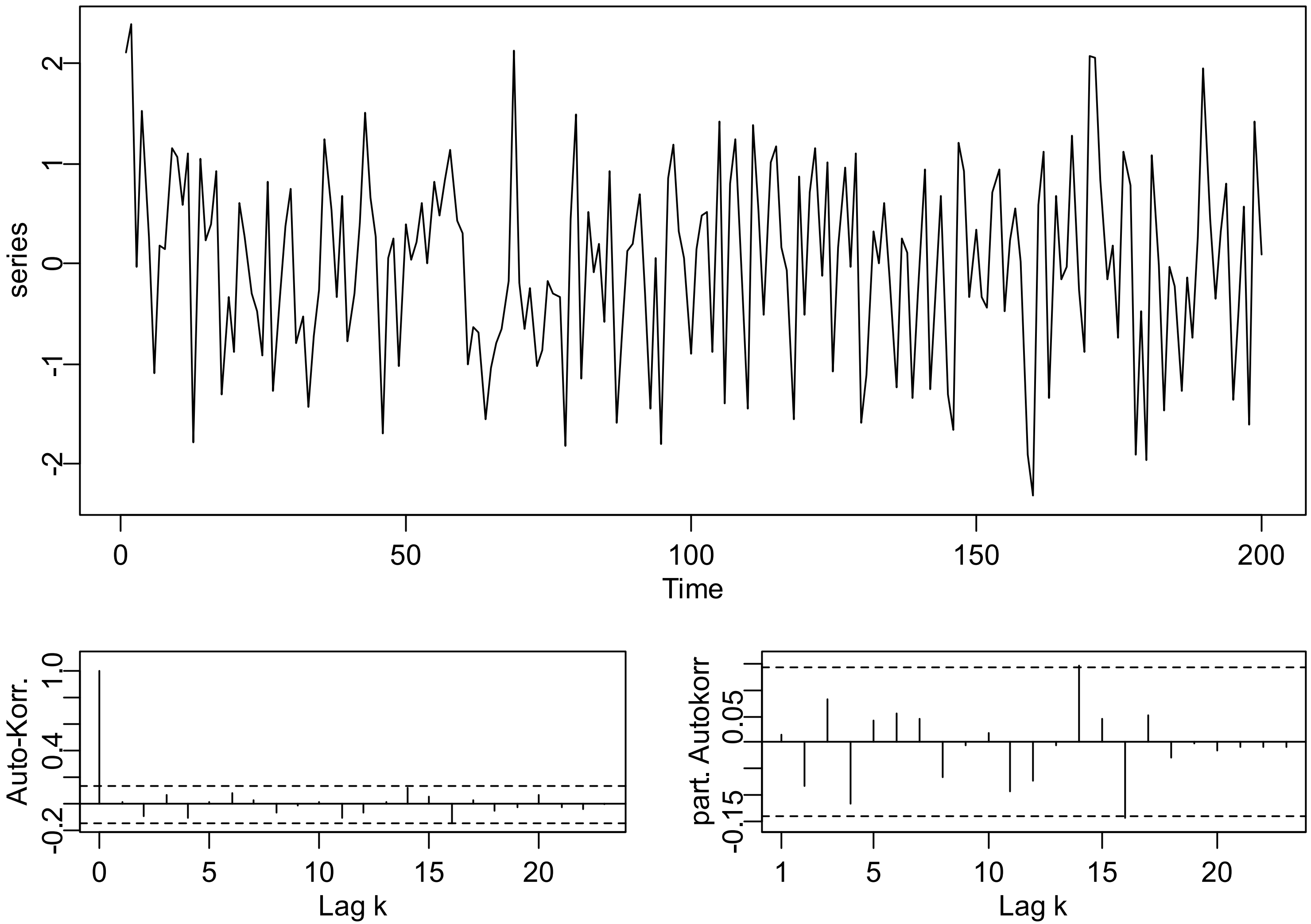
The Series, ACF and PACF



Model 1: AR(4)

```
> fit1

Call: arima(x = my.ts, order = c(4, 0, 0))

Coefficients:

         ar1      ar2     ar3      ar4  intercept
      1.5430  -1.2310  0.7284  -0.3000     0.6197
s.e.  0.0676   0.1189  0.1189   0.0697     0.2573

sigma^2=0.8923,  log likelihood=-273.67,  aic=559.33
```

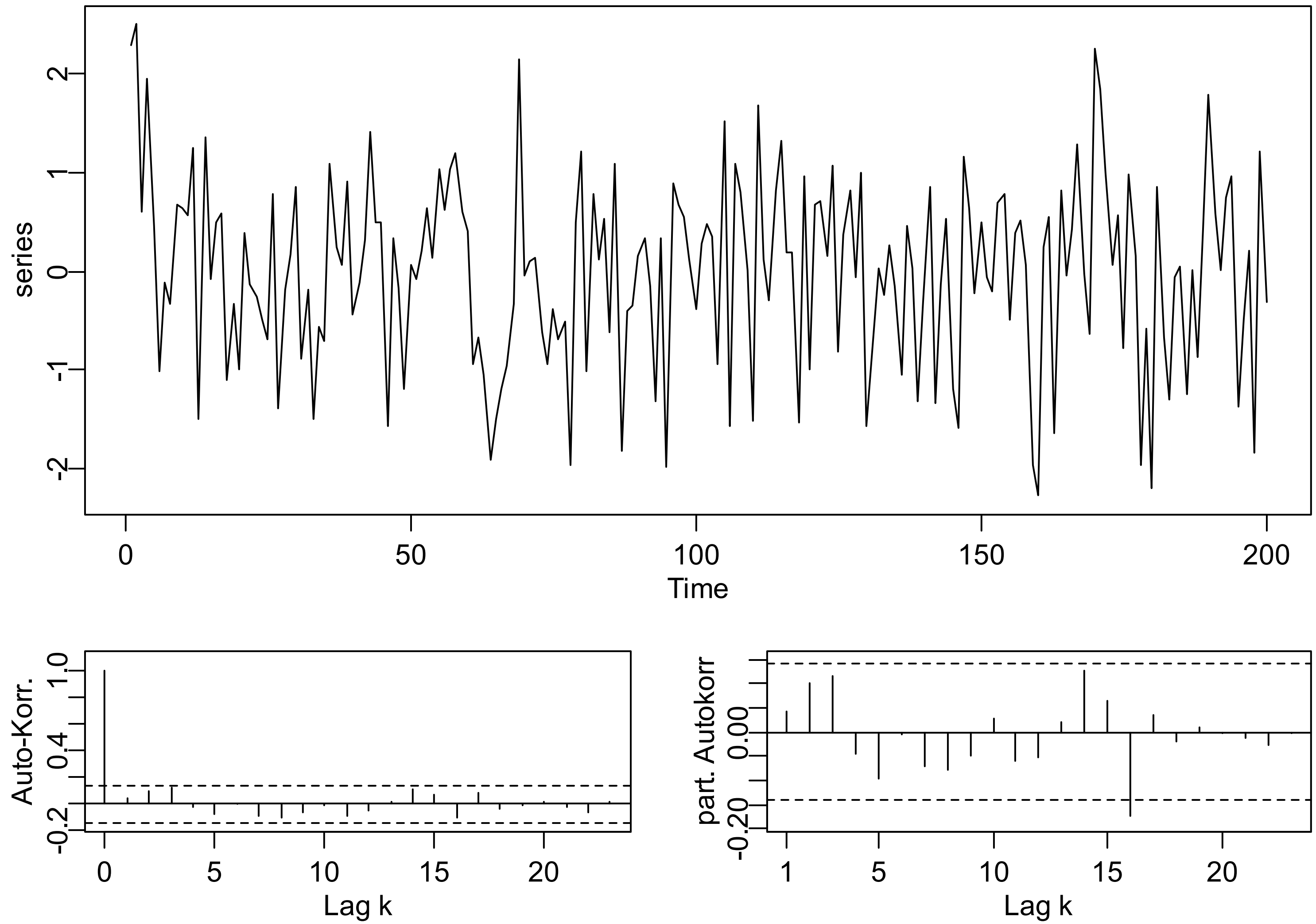
Residuals of Model 1: AR(4)



Model 2: MA(3)

```
> fit2

Call: arima(x = my.ts, order = c(0, 0, 3))

Coefficients:

         ma1     ma2     ma3  intercept
      1.5711  1.0056  0.3057     0.6359
s.e.  0.0662  0.0966  0.0615     0.2604

sigma^2=0.9098,  log likelihood=-275.64,  aic=561.29
```

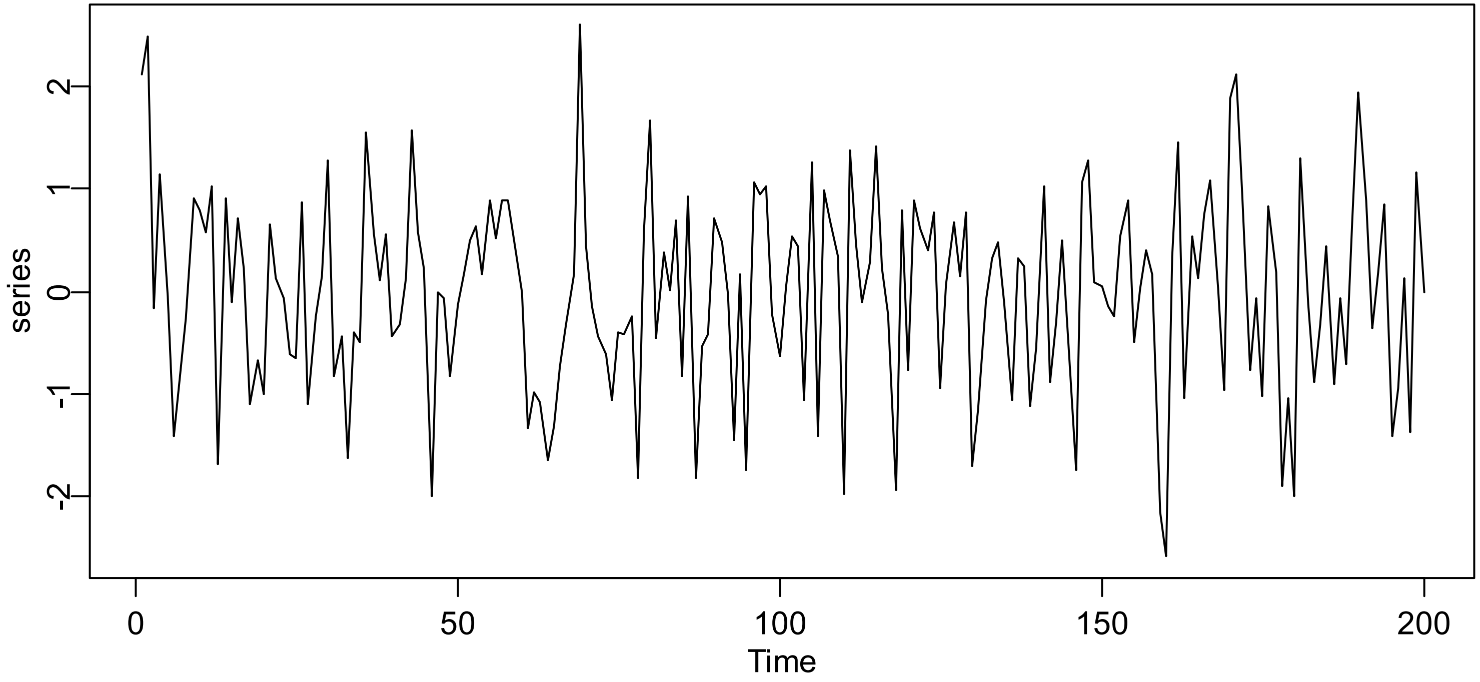
Residuals of Model 2: MA(3)

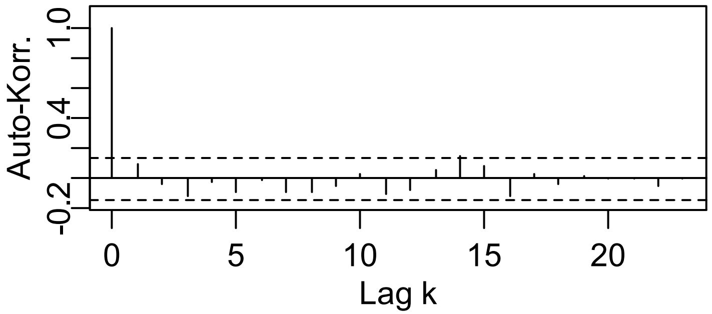


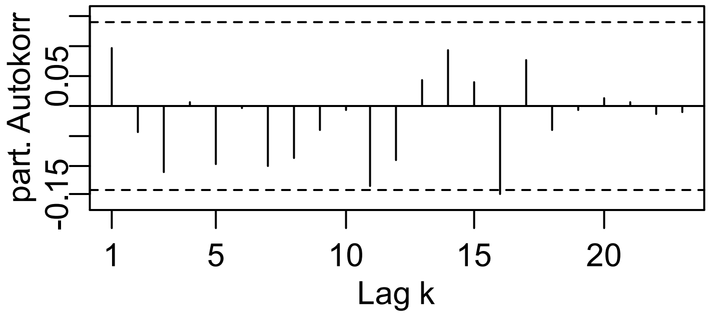
Model 3: ARMA(1,1)

```
> fit3

Call: arima(x = my.ts, order = c(1, 0, 1))

Coefficients:

         ar1     ma1  intercept
      0.6965  0.7981     0.6674
s.e.  0.0521  0.0400     0.3945

sigma^2=0.9107, log likelihood=-275.72,  aic=559.43
```

Residuals of Model 3: ARMA(1,1)







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Model 4: ARMA(2,1)

```
> fit4

Call: arima(x = my.ts, order = c(2, 0, 1))

Coefficients:

         ar1      ar2     ma1  intercept

      0.8915  -0.2411  0.7061     0.6420

s.e.  0.0855   0.0856  0.0625     0.3208

sigma^2=0.8772, log likelihood=-272.01, aic=554.02
```

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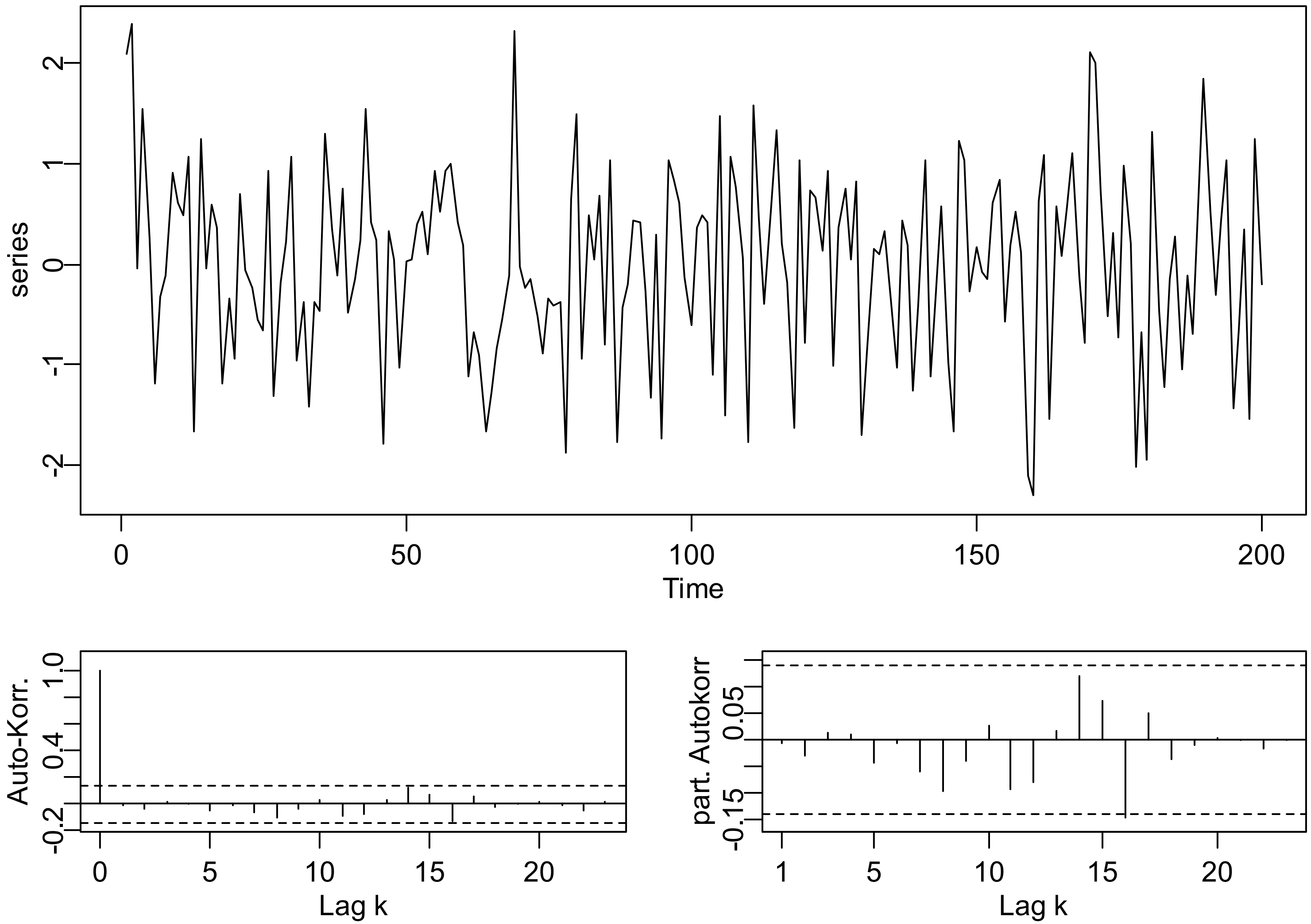
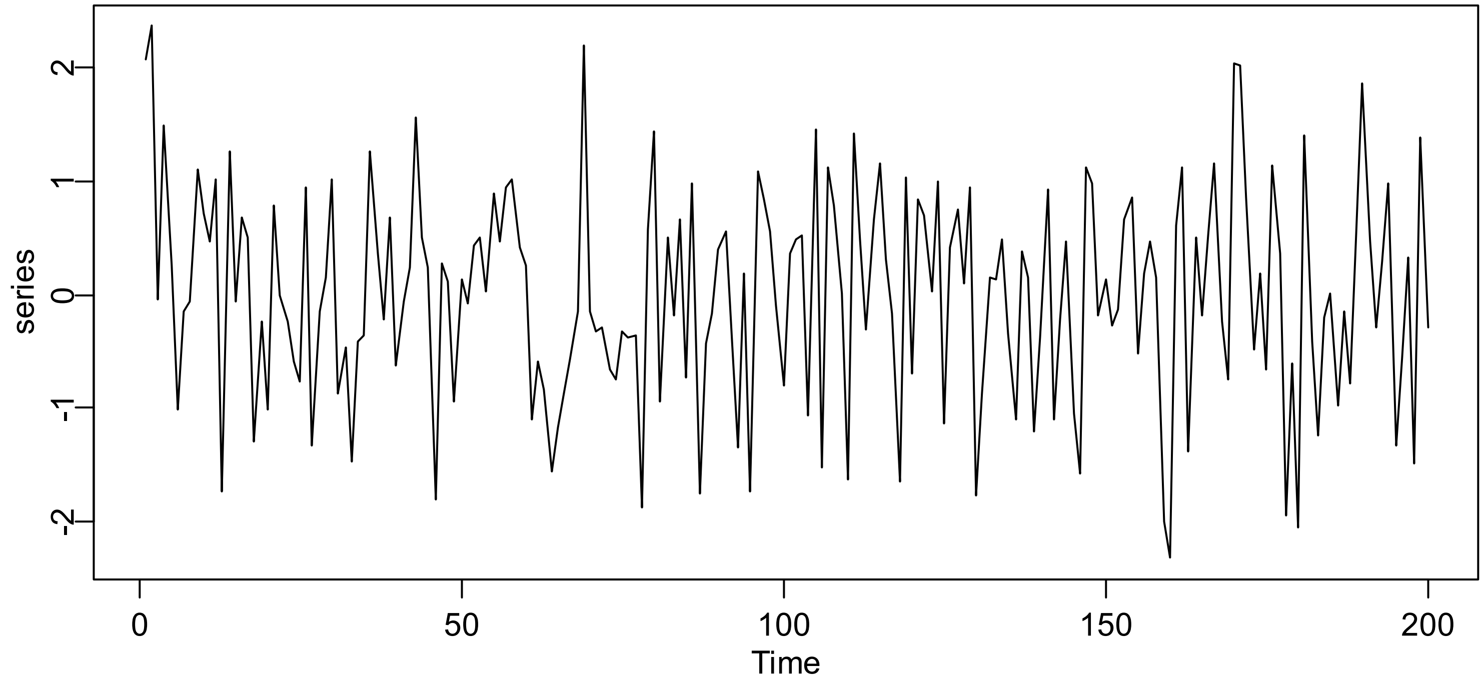
Residuals of Model 4: ARMA(2,1)

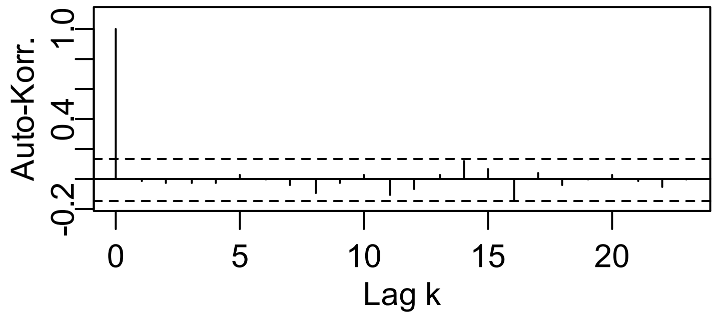


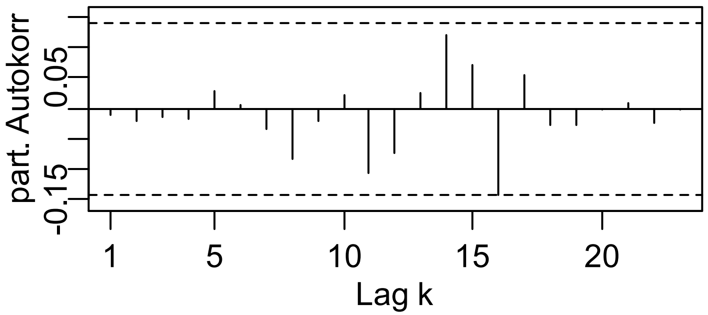
Model 5: ARMA(4,1)

```
> fit5

Call: arima(x = my.ts, order = c(4, 0, 1))

Coefficients:

         ar1      ar2     ar3      ar4     ma1  intercept
      1.0253  -0.4693  0.2190  -0.1280  0.5733     0.6312
s.e.  0.1725   0.2658  0.2124   0.1062  0.1653     0.2930

sigma^2=0.8708, log likelihood=-271.3, aic = 556.59
```

Residuals of Model 5: ARMA(4,1)







Summary of the Order Choice Problem

- Regarding ACF/PACF, all 5 models are plausible
  → ARMA(2,1) would be my favorite

- The residuals look fine (i.e. independent) for all 5 models
  → no further evidence for a particular model

- Regarding AIC, the ARMA models do better
  → ARMA(2,1) would be my favorite

- Significance of the coefficients
  → excludes the ARMA(4,1) as the last contender

Best choice: ARMA (2,1)