Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

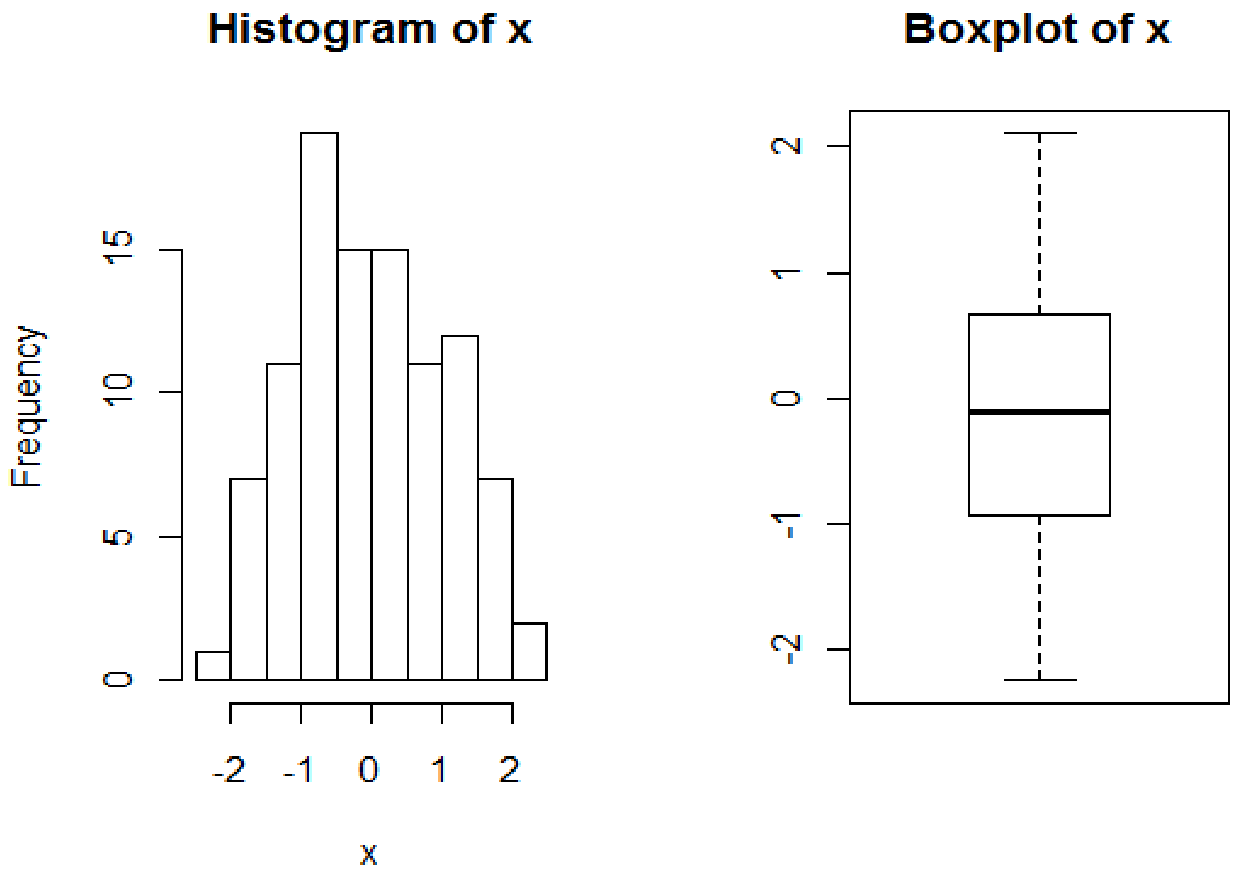
Visualization 1

Applied Multivariate Statistics – Spring 2012

Goals

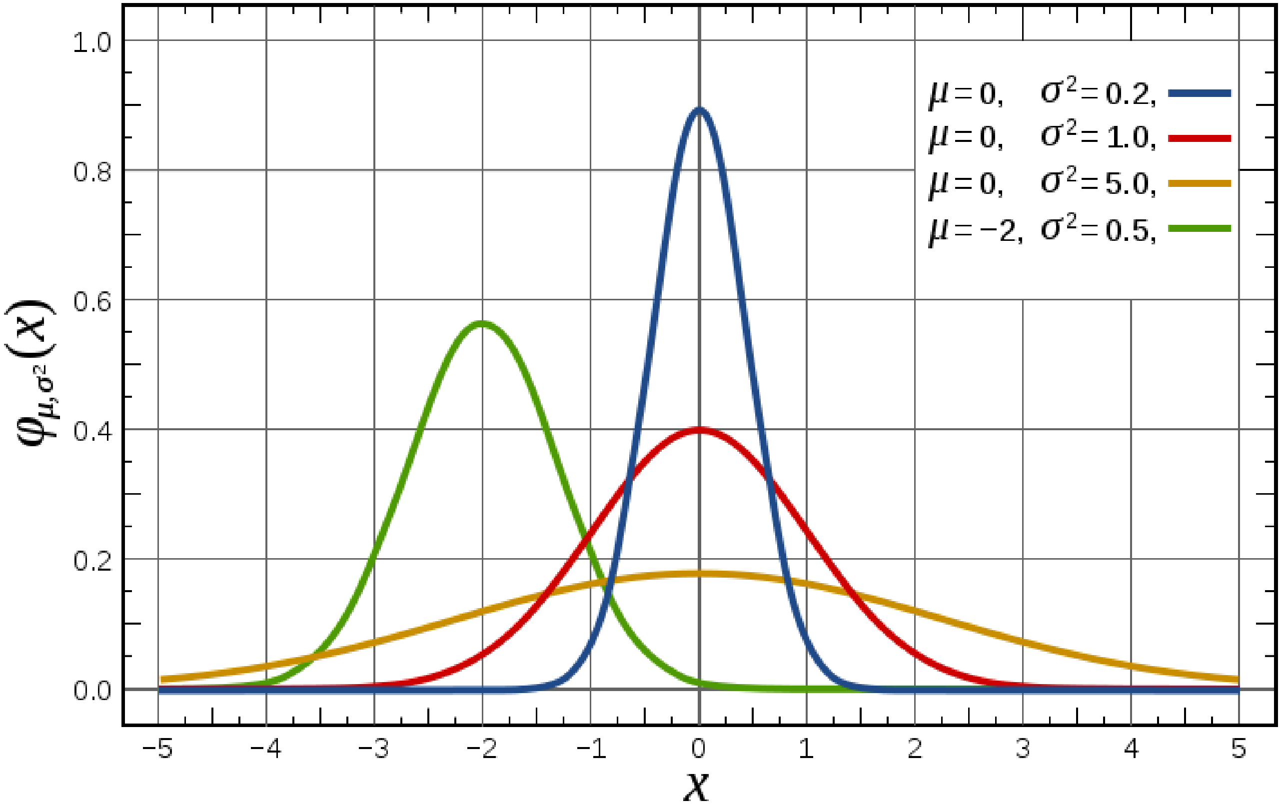
- Covariance, Correlation (true / sample version)
- Test for zero correlation: Fisher's z-Transformation
- Scatterplot / Scatterplotmatrix
- Covariance matrix / Correlation matrix
- Multivariate Normal Distribution
- Mahalanobis distance

Visualization in 1d



Normaldistribution in 1d: Most common model choice

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}\right)$$



Squared Mahalanobis Distance
=
Sq. Distance from mean in standard deviations

Normaldistribution in 1d:
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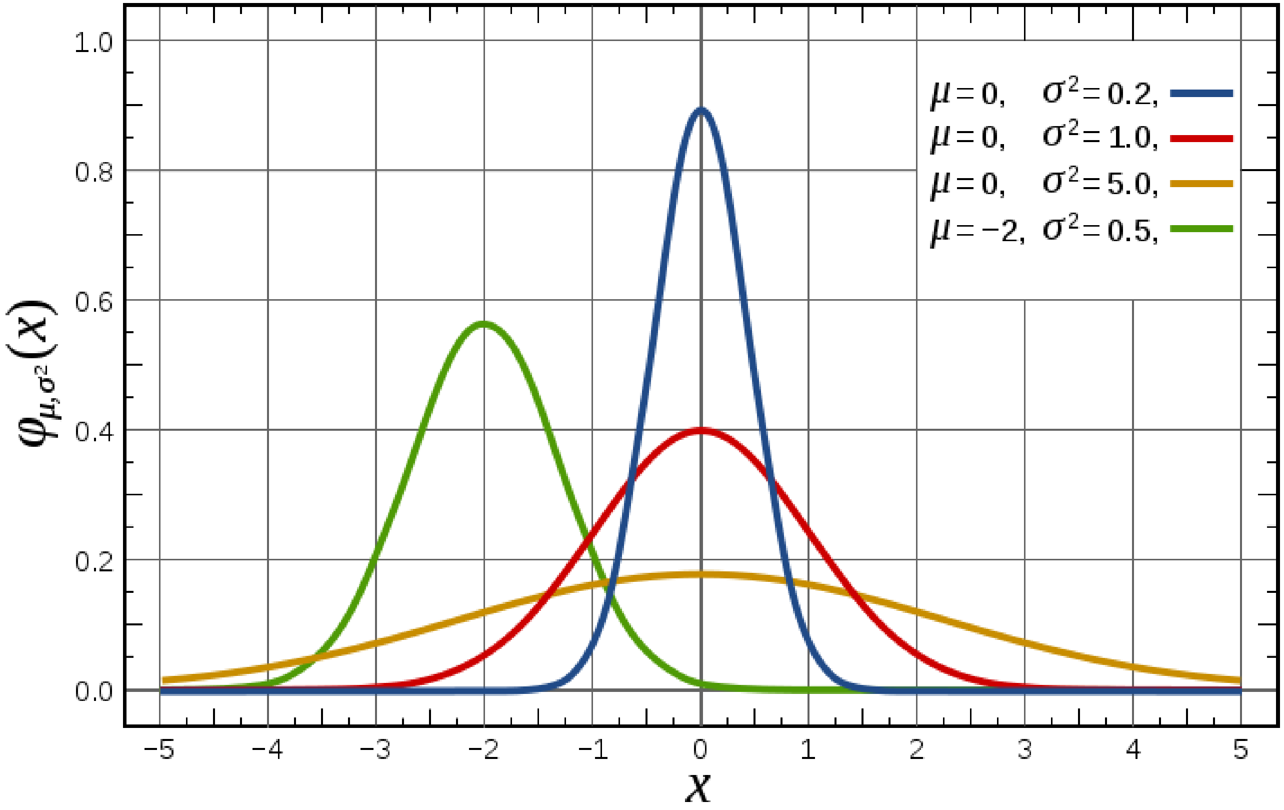
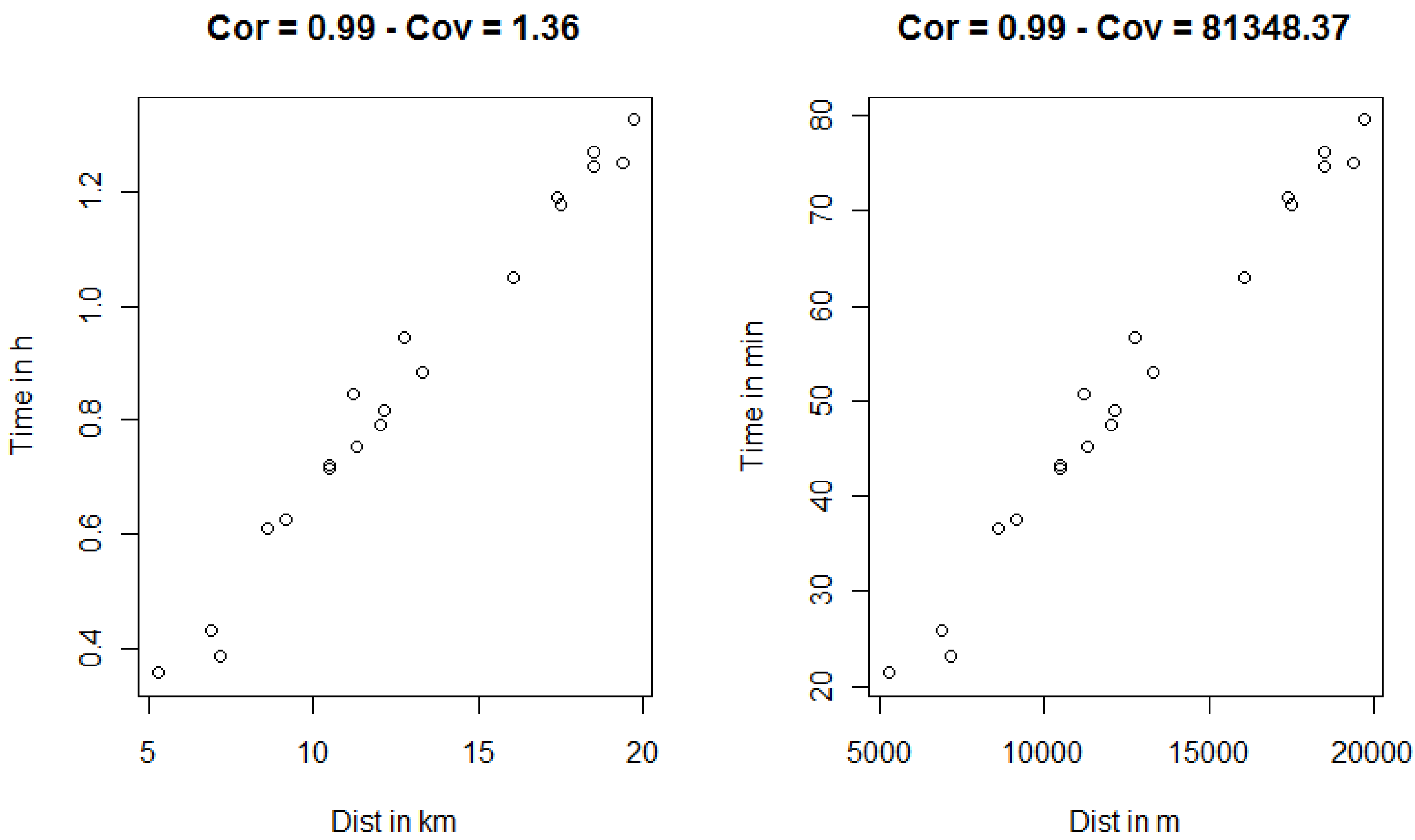


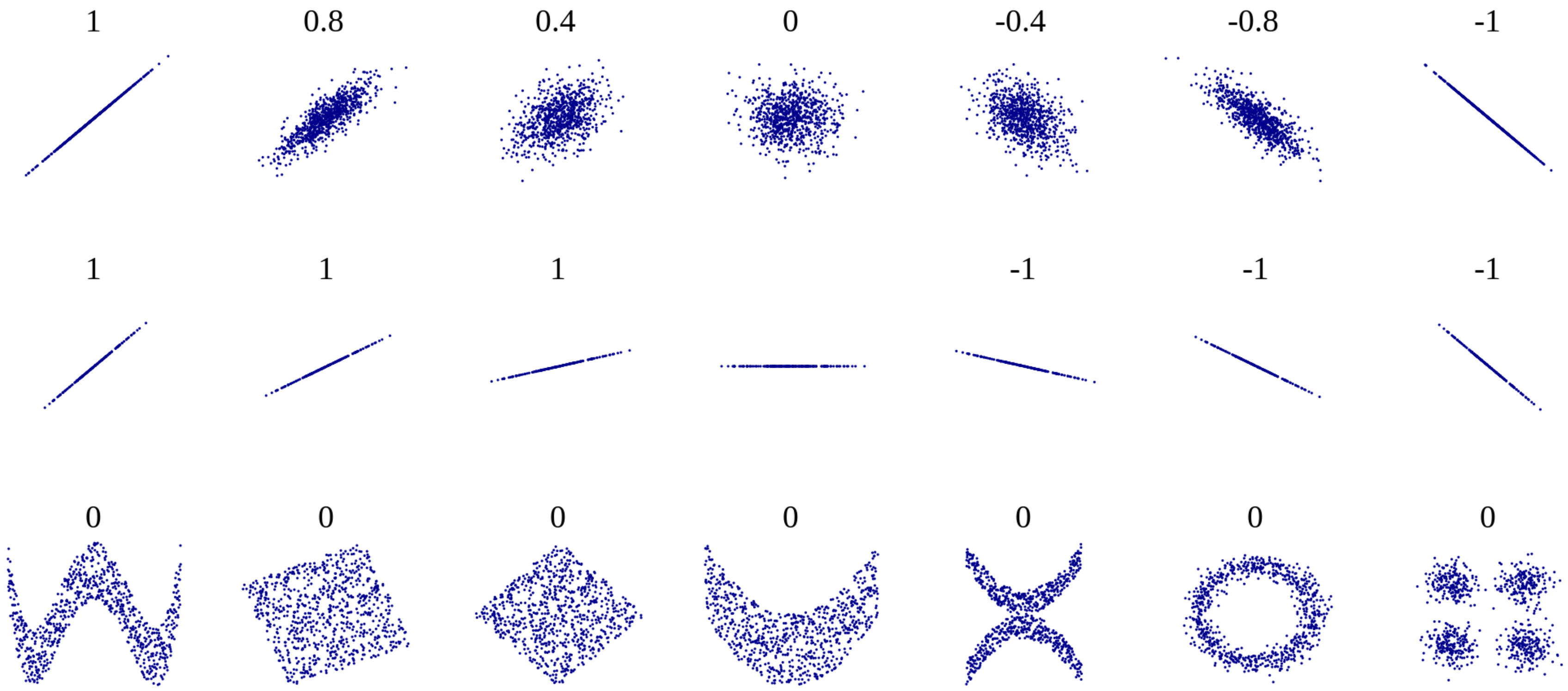
Two variables: Covariance and Correlation

- Covariance: $Cov(X,Y) = E[(X-E[X])(Y-E[Y])] \quad \in [-\infty; \infty]$
- Correlation: $Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \quad \in [-1; 1]$
- Sample covariance: $\widehat{Cov}(x,y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$
- Sample correlation: $r_{xy} = \widehat{Cor}(x,y) = \frac{\widehat{Cov}(x,y)}{\hat{\sigma}_x \hat{\sigma}_y}$
- Correlation is invariant to changes in units, covariance is not (e.g. kilo/gram, meter/kilometer, etc.)

Scatterplot: Correlation is scale invariant



Intuition and pitfalls for correlation
Correlation = LINEAR relation



Test for zero correlation: Fisher's z-Test

- X, Y (bivariate) normal distributed with true correlation ρ
- Take n samples
- Compute sample correlation r

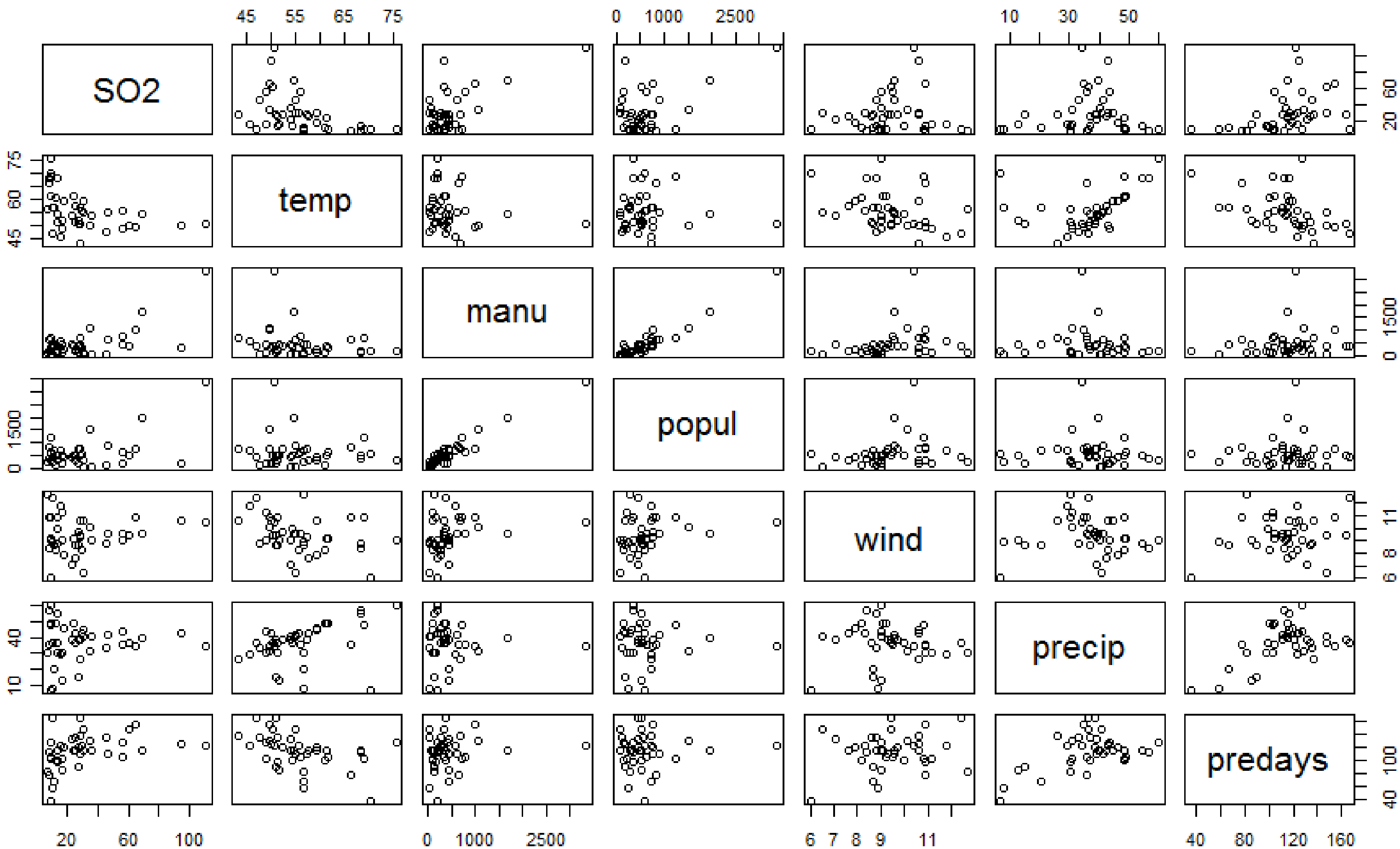
Compute $z = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)$

Compute $\xi = \frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right)$

- For large n: $\sqrt{n-1}(z-\xi) \ \sim \ N(0,1)$

- Use cor.test() in R to test and get confidence intervals

Many dimensions: Scatterplot matrix



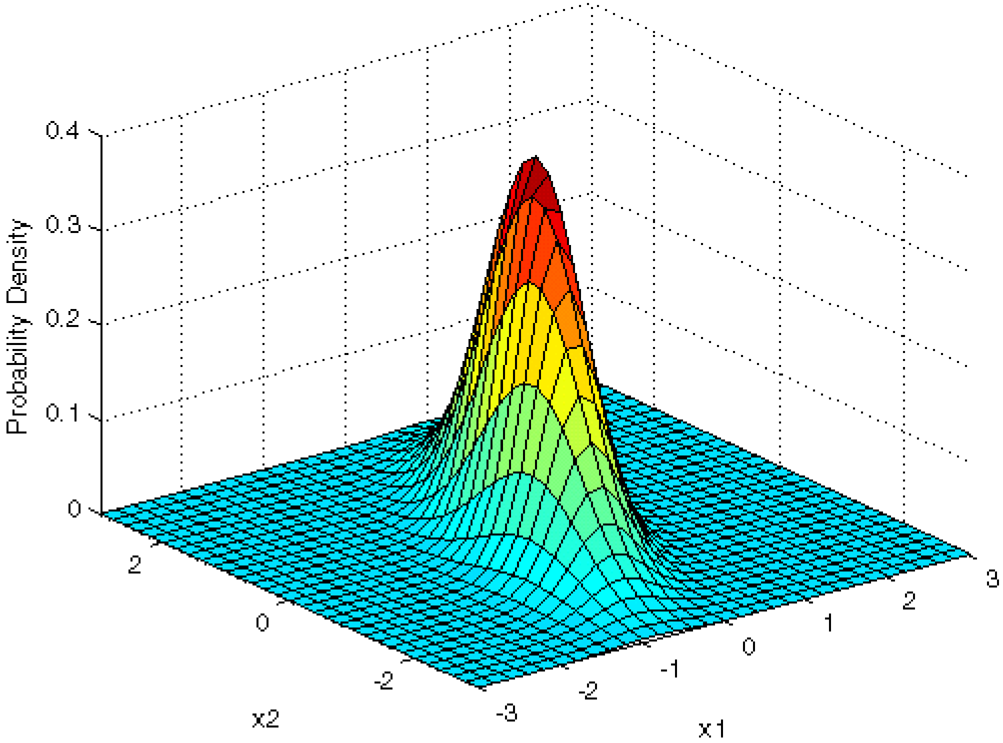
Covariance matrix / correlation matrix: Table of pairwise values

- True covariance matrix: $\Sigma_{ij} = Cov(X_i, X_j)$
- True correlation matrix: $C_{ij} = Cor(X_i, X_j)$

- Sample covariance matrix: $S_{ij} = \widehat{Cov}(x_i, x_j)$
  Diagonal: Variances
- Sample correlation matrix: $R_{ij} = \widehat{Cor}(x_i, x_j)$
  Diagonal: 1

Multivariate Normal Distribution: Most common model choice

$$f(x;\mu,\Sigma) = \frac{1}{\sqrt{2\pi|\Sigma|}}\exp\left(-\frac{1}{2}\cdot(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$



Multivariate Normal Distribution: Funny facts

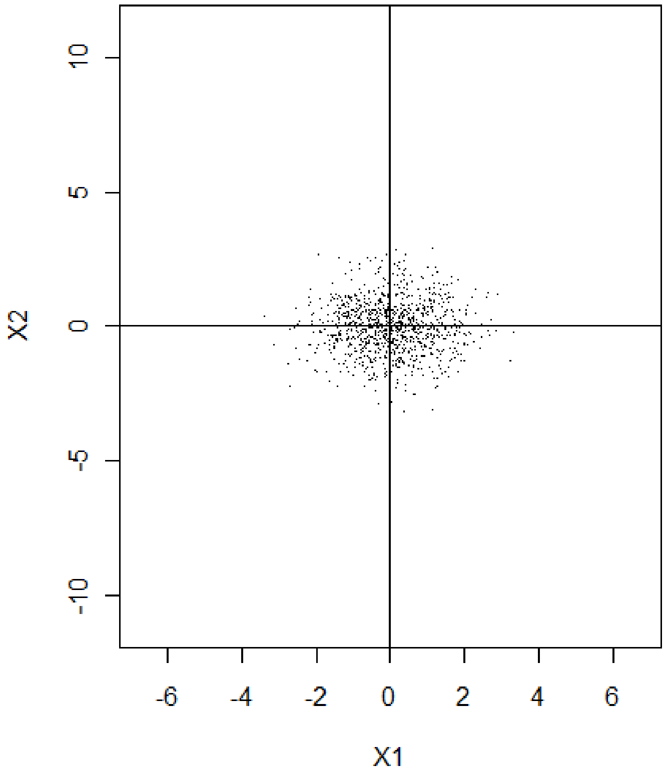
If $X_1, \ldots, X_p$ multivariate normal, then

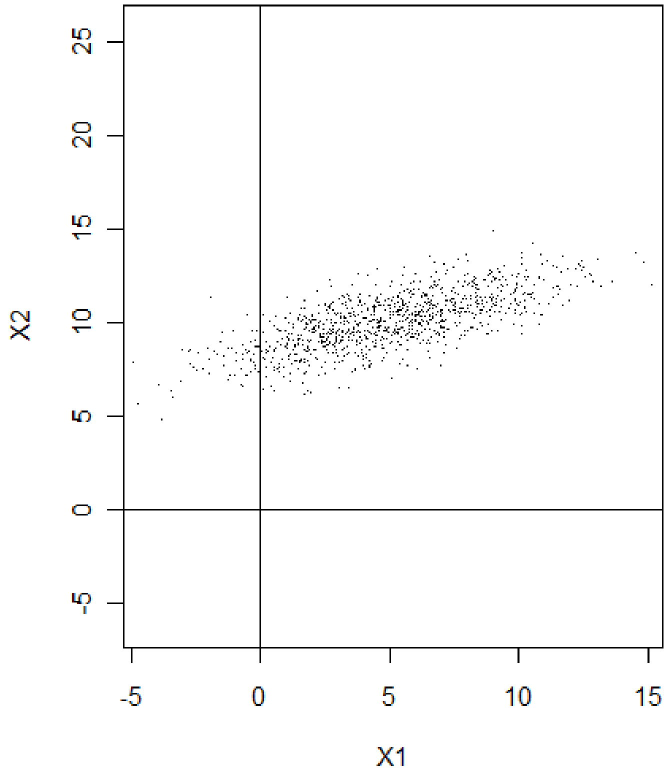
- every linear combination $Y = a_1 X_1 + \ldots + a_p X_p$ is normally distributed
- every projection on a subspace is multivariate normally distributed

If margins are normally distributed, then it is NOT GUARANTEED that the underlying distribution is multivariate normal

(i.e., “multivariate” is stronger than just normal margins)

Multivariate Normal Distribution: Two examples
1000 random samples





$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 5 \\ 10 \end{pmatrix}, \Sigma = \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}$$

Sq. Mahalanobis Distance $MD^2(x)$

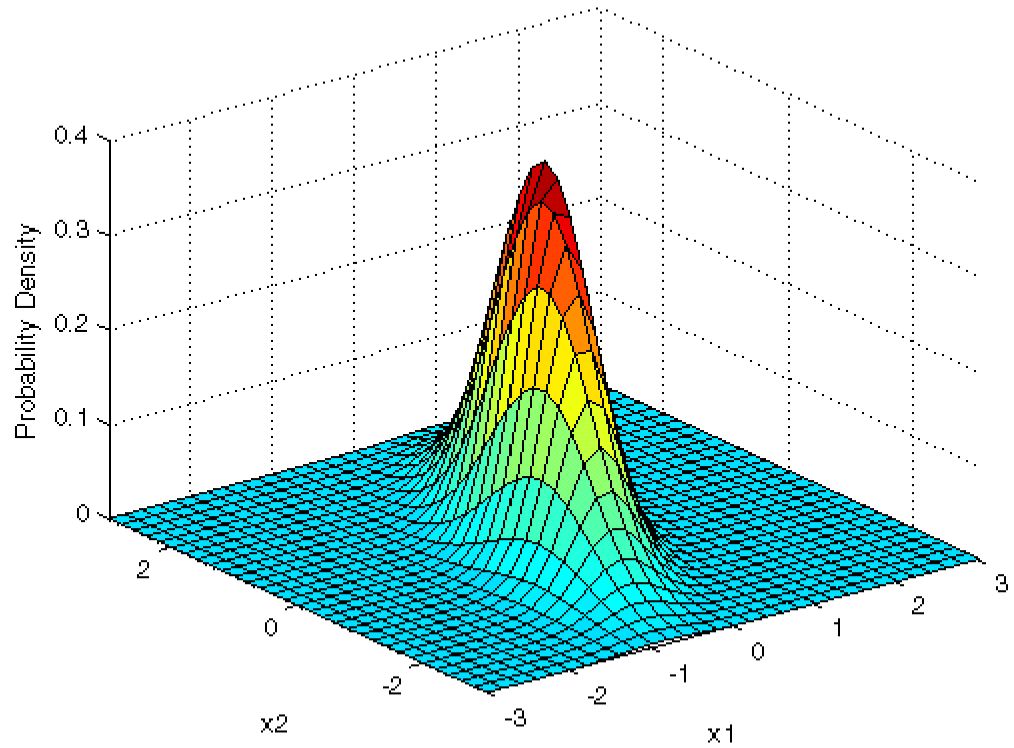
=

Sq. distance from mean in standard deviations

IN DIRECTION OF X

Multivariate Normal Distribution: Most common model choice

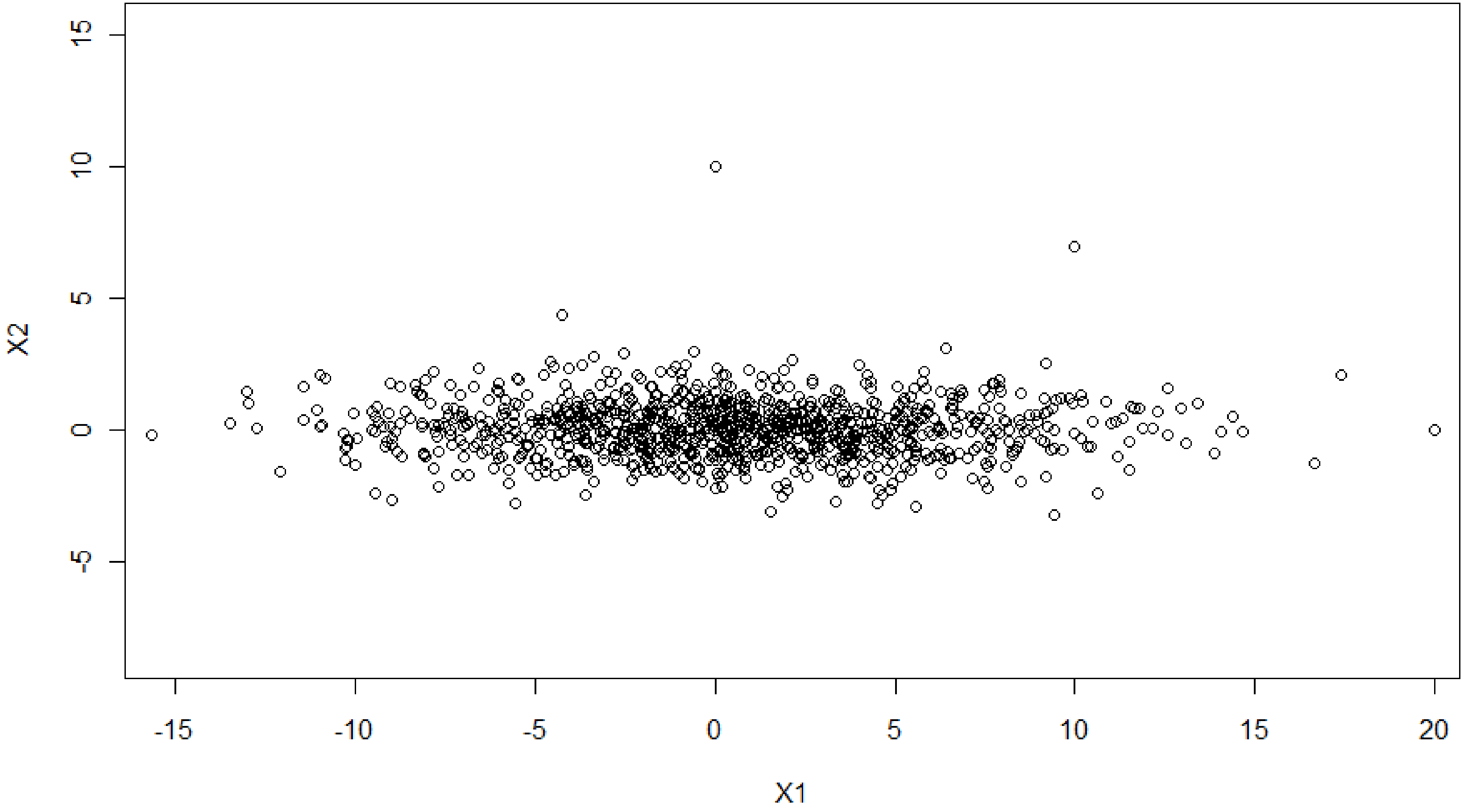
$$f(x;\mu,\Sigma) = \frac{1}{\sqrt{2\pi|\Sigma|}} \exp\left(- \frac{1}{2} \cdot (x-\mu)^T \Sigma^{-1} (x-\mu) \right)$$



$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Mahalanobis distance: Example

$$\Sigma = \begin{pmatrix} 25 & 0 \\ 0 & 1 \end{pmatrix}$$



Mahalanobis distance: Example

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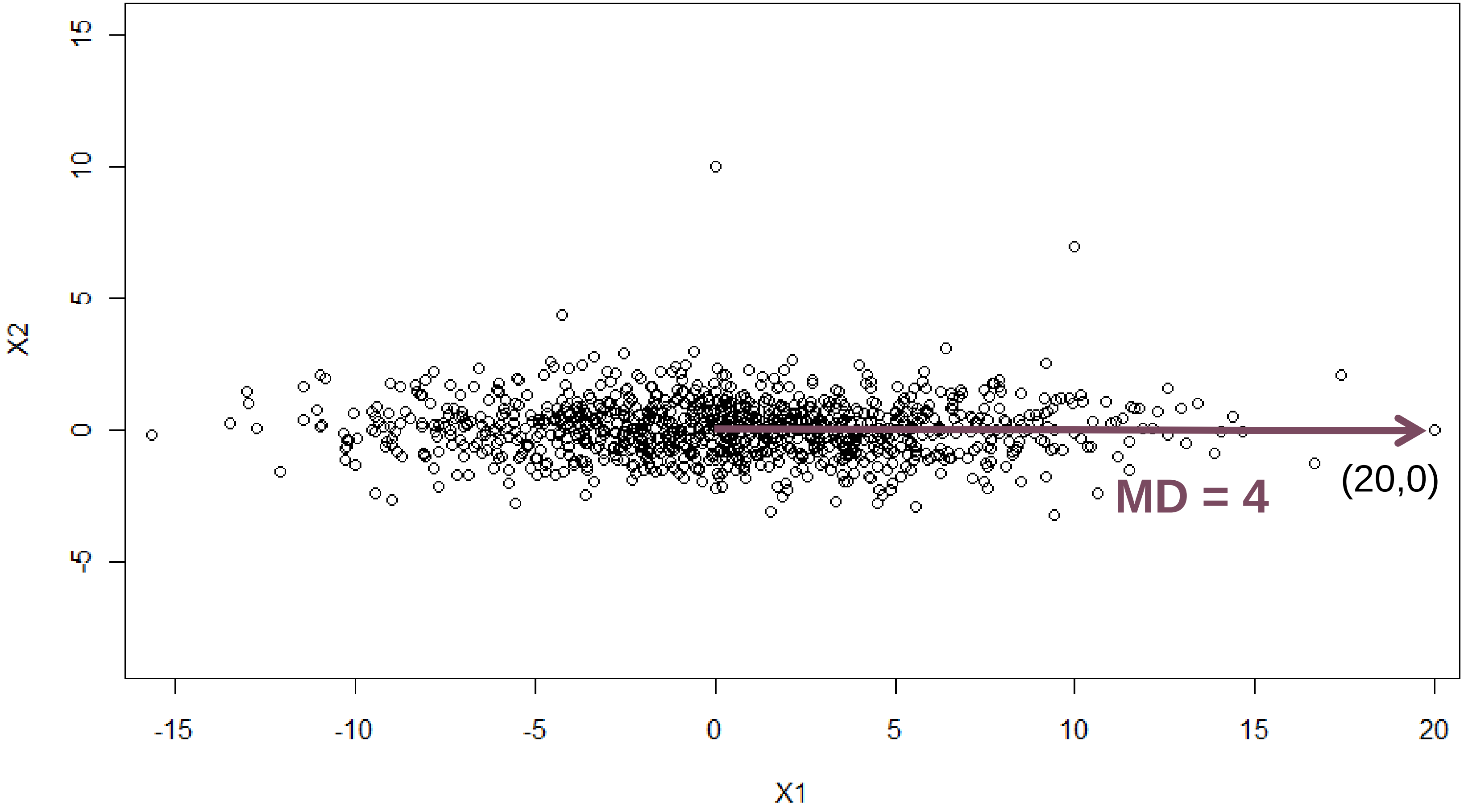

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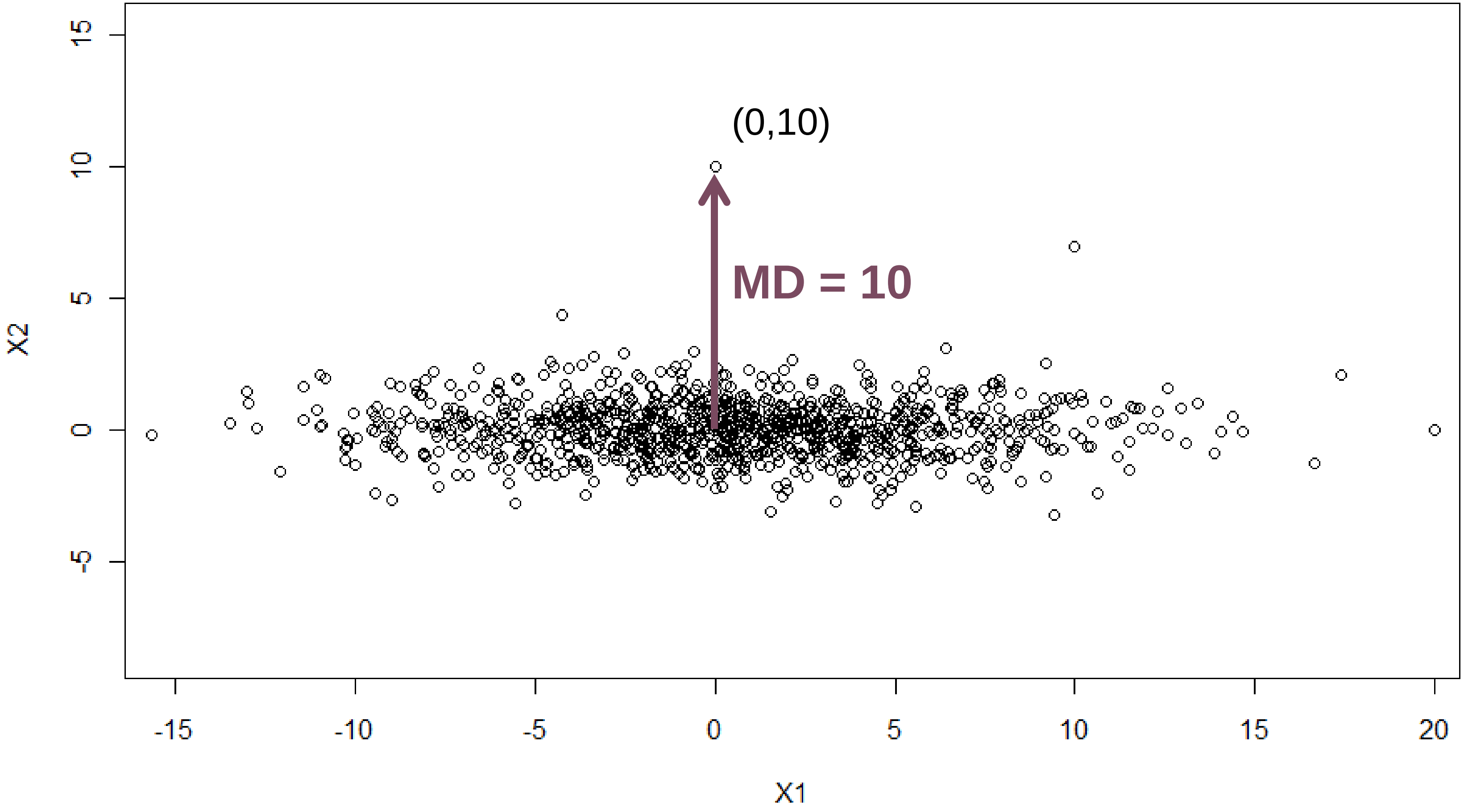

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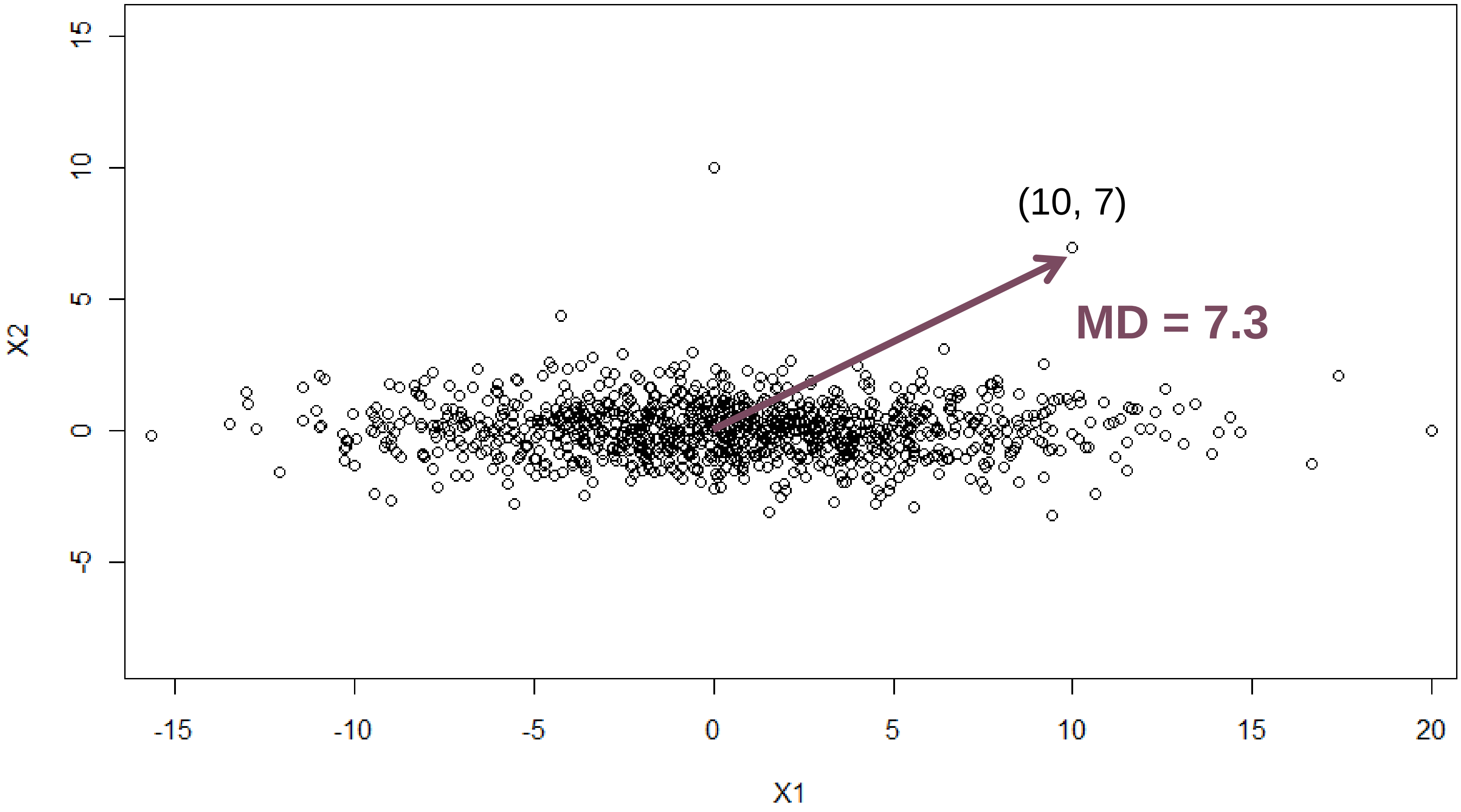

Concepts to know

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- Test for zero correlation: Fisher's z-Transformation
- Scatterplot / Scatterplotmatrix
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R commands to know

- read.csv, head, str, dim
- colMeans, cov, cor
- mvrnorm, t, solve, %*%