Factorial Treatment Structure: Part I

Lukas Meier, Seminar für Statistik

Factorial Treatment Structure

- So far (in CRD), the treatments had **no “structure”**.
- So called **factorial treatment structure** exists if the g treatments are the **combination** of the **levels of two or more factors**.
- In the case that we see all the possible combinations of the levels of the two factors, we call the factors **crossed**.
- Examples
  - Biomass of crop: different **fertilizers** and different **crop varieties**.
  - Battery life: Different **temperature levels** and different **plate material**. (Montgomery, 1991, Example 7-3.1).
  - …

Example (Linder, A. und W. Berchtold, 1982)

- Response: Needleweight of 20 three-week old pine seedlings [in 1/100 g].
- 2 factors: $A =$ “origin”, $B =$ “exposure to light”
- We denote by y_{ijk} kth response of the treatment formed by the ith level of factor A and the jth level of factor B.

$$y_{ijk}$$

Factor A
on level i
(here:*origin*)

Factor B
on level j
(here: *exposure*)

Replicate number
(here: $k = 1, 2$)

Two Factor Design: Generic Data Table

	$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\ldots$
$\boldsymbol{A_1}$	y_{111} y_{112} y_{113} y_{114}	y_{121} y_{122} y_{123} y_{124}	y_{131} y_{132} y_{133} y_{134}	
$\boldsymbol{A_2}$	y_{211} y_{212} y_{213} y_{214}	y_{221} y_{222} y_{223} y_{224}	y_{231} y_{232} y_{233} y_{234}	
$\ldots$	$\ldots$	$\ldots$	$\ldots$	

Data Table of Our Example

	Short	*Long*	*Permanent*
Taglieda	25 25	42 38	62 55
Pfyn	45 42	62 58	80 75
Rheinau	50 50	52 62	88 95

Visualization

- As for one-way ANOVA situation
  - for all treatment combinations
  - factor-wise summaries (“marginal summaries”)
- More useful: **Interaction plot** (see R-code)

Factorial Treatment Structure

- The **structure of the treatment** influences the **analysis** of the data.

- Setup:
  - Factor A with a levels
  - Factor B with b levels
  - n replicates for **every** combination ← a so called **balanced design**
  - Total of $N = a \cdot b \cdot n$ observations

- We could analyze this with the usual **cell means model** (**ignoring** the special treatment structure)

- Typically, we have research questions about **both** factors and their possible **interaction (interplay)**.

Factorial Treatment Structure

- Examples:
  - "Is effect of light exposure location specific?"
    (→ interaction between light exposure and location)
  - "What is the effect of light exposure averaged over all locations?"
    (→ main effect of light exposure)
  - "What is the effect of location averaged over all exposure levels?"
    (→ main effect of location)

- We could use the cell means (one-way ANOVA) model and try to answer these questions with **appropriate contrasts** (→ complicated).

- Easier: Use a model that incorporates the **factorial structure** of the treatments.

Factorial Model

because two factors involved

The **two-way ANOVA model with interaction** is

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

where

- α_i is the **main effect** of factor A at level i.
- β_j is the **main effect** of factor B at level j.
- $(\alpha\beta)_{ij}$ is the **interaction effect** between A and B for level combination i, j (**not** the product $\alpha_i\beta_j$!)
- ϵ_{ijk} are i.i.d. $N(0, \sigma^2)$ **errors**.
- Typically, **sum-to-zero constraints** are being used, i.e.
  - $\sum_{i=1}^{a} \alpha_i = 0$, $\sum_{j=1}^{b} \beta_j = 0$.

    → $a - 1$ and $b - 1$ degrees of freedom
  - $\sum_{i=1}^{a} (\alpha\beta)_{ij} = 0$, $\sum_{j=1}^{b} (\alpha\beta)_{ij} = 0$.

    → $(a-1) \cdot (b-1)$ degrees of freedom

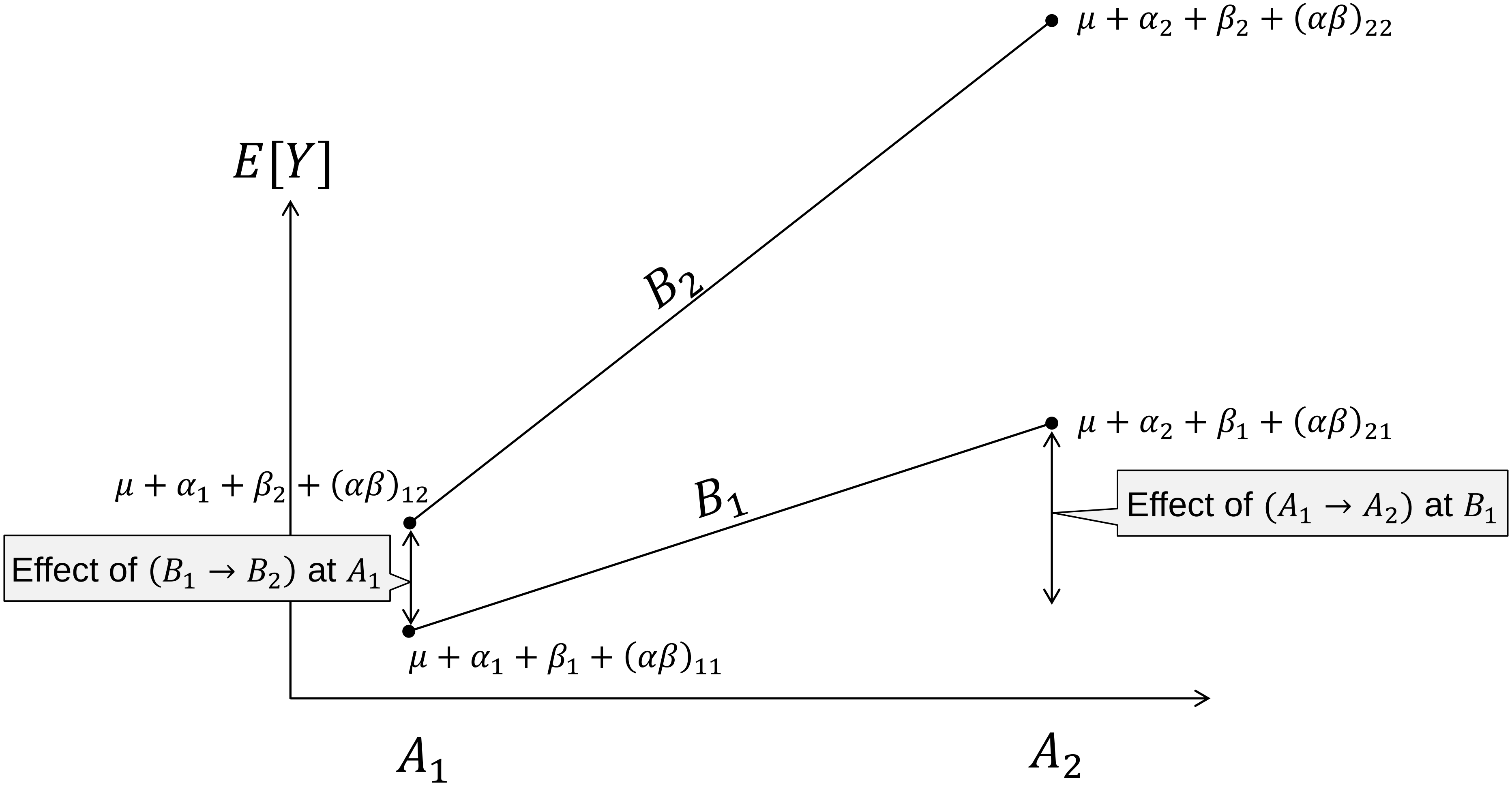
Interpretation of Main Effects

- **Main-effects** are nothing else than the **average effect** when moving from row to row (column to column).
- **Interaction effect** is the difference to the main-effects model, i.e. it measures how far the treatment means differ from the main-effects model.
- If there is **no interaction**, the effects are **additive**. In our example it would mean: “No matter what location we are considering, the effect of light exposure is always the same.”

Visualization of Model

$$E\left[Y_{ijk}\right] = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$

Factor A, Factor B with two levels each

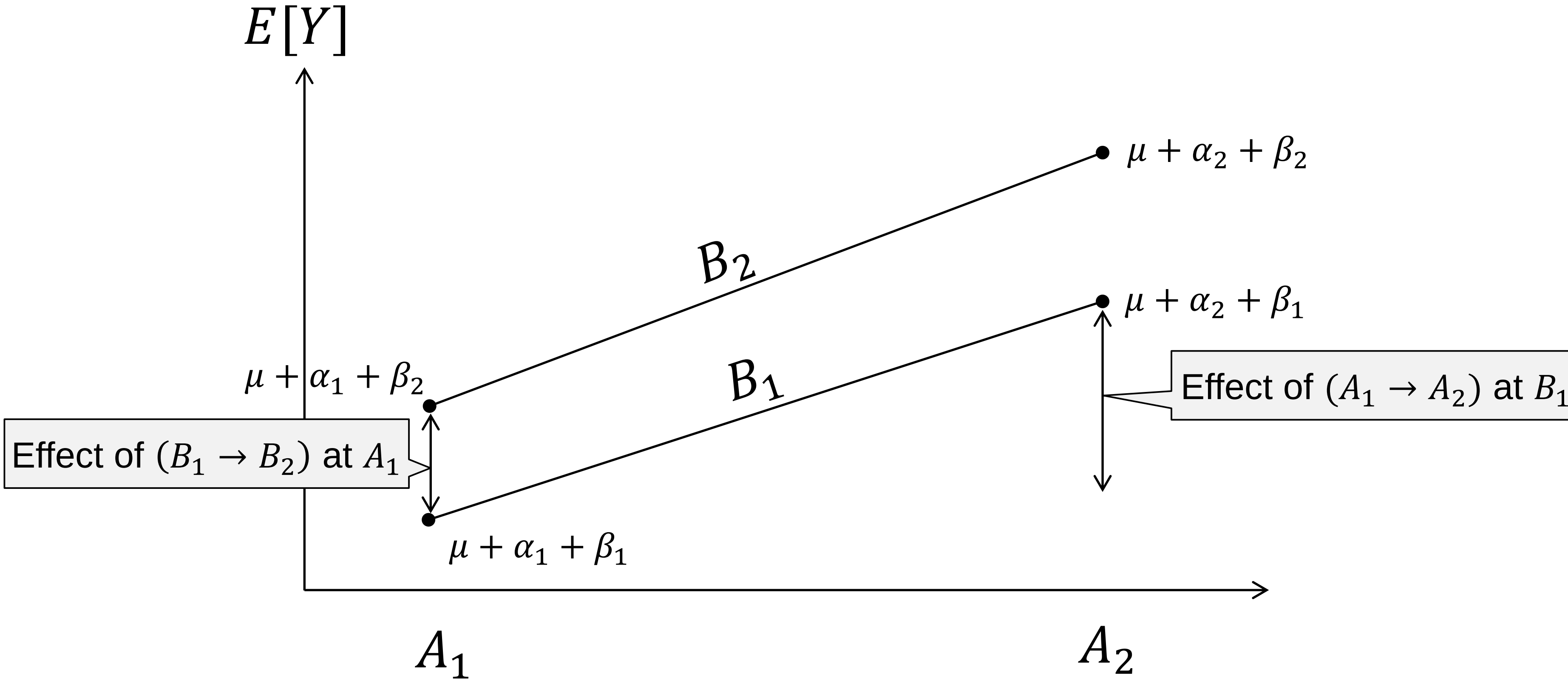


Visualization of Model

$$E[Y_{ijk}] = \mu + \alpha_i + \beta_j$$

If no interaction is present, lines will be **parallel**.

Interaction plot is nothing else than empirical version of this plot.



Two-Way ANOVA Step-By-Step

	Short	Long	Permanent
Taglieda	$\boxed{25}$ 25	42 38	62 55
Pfyn	45 42	62 58	80 75
Rheinau	50 50	52 62	88 95

=

	Short	Long	Permanent
Taglieda	μ μ	μ μ	μ μ
Pfyn	μ μ	μ μ	μ μ
Rheinau	μ μ	μ μ	μ μ

+

	Short	Long	Permanent
Taglieda	α_1 α_1	α_1 α_1	α_1 α_1
Pfyn	α_2 α_2	α_2 α_2	α_2 α_2
Rheinau	α_3 α_3	α_3 α_3	α_3 α_3

+

	Short	Long	Permanent
Taglieda	β_1 β_1	β_2 β_2	β_3 β_3
Pfyn	β_1 β_1	β_2 β_2	β_3 β_3
Rheinau	β_1 β_1	β_2 β_2	β_3 β_3

+

	Short	Long	Permanent
Taglieda	$(\alpha\beta)_{11}$ $(\alpha\beta)_{11}$	$(\alpha\beta)_{12}$ $(\alpha\beta)_{12}$	$(\alpha\beta)_{13}$ $(\alpha\beta)_{13}$
Pfyn	$(\alpha\beta)_{21}$ $(\alpha\beta)_{21}$	$(\alpha\beta)_{22}$ $(\alpha\beta)_{22}$	$(\alpha\beta)_{23}$ $(\alpha\beta)_{23}$
Rheinau	$(\alpha\beta)_{31}$ $(\alpha\beta)_{31}$	$(\alpha\beta)_{32}$ $(\alpha\beta)_{32}$	$(\alpha\beta)_{33}$ $(\alpha\beta)_{33}$

+

	Short	Long	Permanent
Taglieda	$\boxed{\epsilon_{111}}$ ϵ_{112}	ϵ_{121} ϵ_{122}	ϵ_{131} ϵ_{132}
Pfyn	ϵ_{211} ϵ_{212}	ϵ_{221} ϵ_{222}	ϵ_{231} ϵ_{232}
Rheinau	ϵ_{311} ϵ_{312}	ϵ_{321} ϵ_{322}	ϵ_{331} ϵ_{332}

Parameter Estimates

- Estimates for the balanced case (and the sum-to-zero constraints) are

Parameter	Estimator
μ	$\hat{\mu} = \bar{y}_{...}$
α_i	$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}$
β_j	$\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$
$(\alpha\beta)_{ij}$	$\widehat{(\alpha\beta)}_{ij} = \bar{y}_{ij.} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j$

- This means: we estimate the main effects as if the other factors wouldn't be there (see next slides).

Main Effect of Location

- Original data-set

	Short	*Long*	*Permanent*
Taglieda	25 25	42 38	62 55
Pfyn	45 42	62 58	80 75
Rheinau	50 50	52 62	88 95

- Ignore factor "exposure", estimate effect as in one-way ANOVA model

Taglieda	*Pfyn*	*Rheinau*
25 25 42 38 62 55	45 42 62 58 80 75	50 50 52 62 88 95

$\rightarrow \hat{\alpha}_i$

Main Effect of Light Exposure

- Original data-set

	Short	*Long*	*Permanent*
Taglieda	25 25	42 38	62 55
Pfyn	45 42	62 58	80 75
Rheinau	50 50	52 62	88 95

- Ignore factor “location”, estimate effect as in one-way ANOVA model

Short	*Long*	*Permanent*
25	42	62
25	38	55
45	62	80
42	58	75
50	52	88
50	62	95

$\rightarrow \hat{\beta}_j$

Sum of Squares

- Again, total sum of squares can be **partitioned** into **different sources,** i.e. $SS_T = SS_A + SS_B + SS_{AB} + SS_E$.

squared effect

Source	df	Sum of squares (SS)
A	$a-1$	$\sum_{i=1}^{a} \underbrace{b \cdot n}_{\text{\# observations with that effect}} \cdot\ \hat{\alpha}_i^2$
B	$b-1$	$\sum_{j=1}^{b} a \cdot n \cdot \hat{\beta}_j^2$
AB	$(a-1)\cdot(b-1)$	$\sum_{i=1}^{a} \sum_{j=1}^{b} n \cdot \widehat{(\alpha\beta)}_{ij}^2$
Error	$(n-1)\cdot ab$	$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} \left(y_{ijk} - \bar{y}_{ij\cdot}\right)^2$
Total	$abn-1$	$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} \left(y_{ijk} - \bar{y}_{\cdots}\right)^2$

product of *df's* of involved factors

$\underbrace{\#observations - 1}_{\text{degrees of freedom of total}} - \mathrm{sum}(df\ above)$

ANOVA Table

- As before, we can construct an **ANOVA table**

Source	df	SS	MS	F
A	$a-1$	SS_A	$\frac{SS_A}{a-1}$	$\frac{MS_A}{MS_E}$
B	$b-1$	SS_B	$\frac{SS_B}{b-1}$	$\frac{MS_B}{MS_E}$
AB	$(a-1)\cdot(b-1)$	SS_{AB}	$\frac{SS_{AB}}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
$Error$	$ab\cdot(n-1)$	SS_E	$\frac{SS_E}{(n-1)ab}$	

- Under the corresponding (global) null-hypothesis, the F-ratio is again F-distributed with degrees of freedom defined through the numerator / denominator.

F-Tests: Overview

- **Interaction $\boldsymbol{AB}$**
  - $H_0: (\alpha\beta)_{ij} = 0$ for **all** i, j
  - H_A: At least one $(\alpha\beta)_{ij} \neq 0$
  - Under H_0: $\frac{MS_{AB}}{MS_E} \sim F_{(a-1)(b-1),\, ab(n-1)}$
- **Main effect $\boldsymbol{A}$**
  - $H_0: \alpha_i = 0$ for **all** i
  - H_A: At least one $\alpha_i \neq 0$
  - Under H_0: $\frac{MS_A}{MS_E} \sim F_{(a-1),\, ab(n-1)}$
- **Main effect $\boldsymbol{B}$**
  - $H_0: \beta_j = 0$ for **all** j
  - H_A: At least one $\beta_j \neq 0$
  - Under H_0: $\frac{MS_B}{MS_E} \sim F_{(b-1),\, ab(n-1)}$

Analyzing the ANOVA Table

- Typically, the F-tests are analyzed from **bottom to top** (in the ANOVA table).

- Here, this means we **start with the F-test of the interaction.**

- If we reject:
  - Conclude that we need an interaction in our model, i.e. the effect of A depends on the level of B.
  - **Don't continue** testing the main-effects (**principle of hierarchy**)
  - Perform individual analysis for every level of factor A (or B).
  - Have a look at **interaction plot** to get a better understanding of what is going on.
  - Interaction might be based on a **single** cell.
  - Transformation might help in getting rid of the interaction.

- If we can't reject, continue testing the main effects.

Two-Way ANOVA model in R

- Define model with interaction in formula interface

```
> fit <- aov(y ~ location * exposure, data = data)
> summary(fit)
                  Df Sum Sq Mean Sq F value   Pr(>F)    
location           2   2053  1026.4  69.981 3.28e-06 ***
exposure           2   4074  2037.1 138.890 1.72e-07 ***
location:exposure  4    183    45.7   3.117   0.0722 .  
Residuals          9    132    14.7                     
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

More than Two Factors

- Model can be easily extended to **more than two** factors, say 3 (A, B, C).
- Model:

$$
\begin{aligned}
Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + & \qquad \leftarrow \boxed{\text{main effects}} \\
(\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + & \qquad \leftarrow \boxed{\text{two-way interactions}} \\
(\alpha\beta\gamma)_{ijk} + & \qquad \leftarrow \boxed{\text{three-way interaction}} \\
\epsilon_{ijkl} &
\end{aligned}
$$

  with ϵ_{ijkl} i.i.d. $N(0, \sigma^2)$ and the usual sum-to-zero constraints.
- Two-way interaction describes how a main-effect depends on the level of the other factor.
- Three-way interaction describes how a two-way interaction depends on level of third factor…

Interpretation

- The higher-order interactions are quite **difficult to interpret**.
- Parameter estimation as before.
- Degrees of freedom of an interaction is the **product of the df’s of the involved factors**.
- E.g., for the three-way interaction:

$$(a-1)\cdot(b-1)\cdot(c-1)$$