Block Designs

Randomized Complete Block Design

- RCBD is the most widely used experimental design
- More efficient than the 1-factor design
- What is new?
    - Random or fixed effects
    - Correlated observations

Biochemical Experiment

Serum levels after four medical treatments. Only four people can be treated per day, one for each medication.

	Day							
	1	2	3	4	5	6	7	8
Treat.								
I	4.4	5.3	5.3	1.8	3.7	6.5	5.4	5.2
II	2.8	3.3	7.0	2.6	5.9	5.4	6.9	6.8
III	4.8	1.9	4.3	3.1	6.2	5.7	6.2	7.9
IV	6.8	8.7	7.2	4.8	5.1	6.7	9.3	7.9

Block Design

Subjects

Randomisation

$\swarrow \downarrow \searrow$

	Block 1	Block 2	...	Block J
Group 1	$\times$	$\times$		$\times$
Group 2	$\times$	$\times$		$\times$
Group 3	$\times$	$\times$		$\times$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Group I	$\times$	$\times$		$\times$

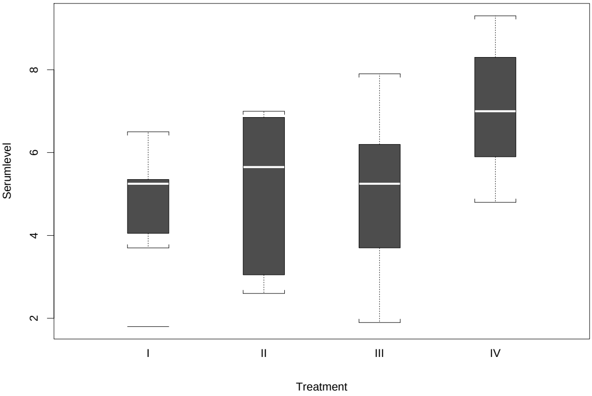
Block Randomisation

R: sample(rep(1:8,4)), sample(4) or sample(32)

Subjects

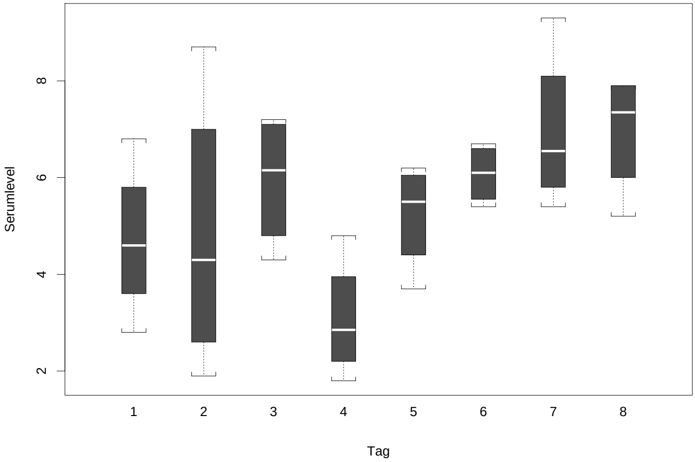
	Day							
Treatment	1	2	3	4	5	6	7	8
I	13	3	26	23	4	28	20	21
II	24	18	6	10	9	25	32	1
III	19	7	8	22	27	30	16	14
IV	2	11	15	12	31	17	29	5

Serum levels by Treatment



Mean:	4.7	5.09	5.01	7.06

Serum levels by Day



Mean: 4.7 4.8 5.95 3.08 5.23 6.07 6.95 6.95

Randomized Complete Block Design

- Each treatment in each block equally often.
- Model:

$$Y_{ij} = \mu + A_i + b_j + \epsilon_{ij} \qquad i = 1, \ldots, I; j = 1, \ldots, J$$

  b_j: Effect of block j
- Fixed-Effects Model:
  $\sum A_i = 0,\ \sum b_j = 0,\ \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$

  Mixed Model:
  $\sum A_i = 0,\ b_j \sim \mathcal{N}(0, \sigma_b^2),\ \epsilon_{ij} \sim \mathcal{N}(0, \sigma_e^2)$
  all b_j and ϵ_{ij} independent.

Anova table

$$y_{ij} - y_{..} = \underbrace{y_{i.} - y_{..}}_{\text{deviation of the treatment mean}} + \underbrace{y_{.j} - y_{..}}_{\text{deviation of the block mean}} + \underbrace{y_{ij} - y_{i.} - y_{.j} + y_{..}}_{\text{residual}}$$

$$SS_{tot} = SS_{treat} + SS_{blocks} + SS_{res}$$

Source	SS	df	MS	F
Blocks	47.3	$J-1=7$	6.75	
Treatments	27.9	$I-1=3$	9.29	...
Residual	35.3	$(I-1)(J-1)=21$	1.68	
Total	110.4	$N-1=31$		

Expected mean squares

Fixed-effects model

$$\begin{aligned} E(MS_{res}) &= \sigma^2 \\ E(MS_{treat}) &= \sigma^2 + J \sum A_i^2/(I-1) \\ E(MS_{block}) &= \sigma^2 + I \sum b_j^2/(J-1) \end{aligned}$$

Mixed-effects model

$$\begin{aligned} E(MS_{res}) &= \sigma_e^2 \\ E(MS_{treat}) &= \sigma_e^2 + J \sum A_i^2/(I-1) \\ E(MS_{block}) &= \sigma_e^2 + I\sigma_b^2 \end{aligned}$$

F Tests

Fixed-effects Model

$$H_0 : A_i = 0 \quad \forall i, \qquad F = \frac{MS_{treat}}{MS_{res}} \sim F_{I-1,(I-1)(J-1)}$$
$$H_0 : b_j = 0 \quad \forall j, \qquad F = \frac{MS_{blocks}}{MS_{res}} \sim F_{J-1,(I-1)(J-1)}$$

Mixed Model

$$H_0 : A_i = 0 \quad \forall i, \qquad F = \frac{MS_{treat}}{MS_{res}} \text{ as above}$$
$$H_0 : \sigma_b^2 = 0 \qquad F = \frac{MS_{blocks}}{MS_{res}}$$

Blocks are usually not tested
$MS_{blocks} \gg MS_{res}$: Blocking good
$MS_{blocks} \leq MS_{res}$: Blocking not necessary

Biochemical example

Source	SS	df	MS	F	P value
Blocks	47.3	7	6.75		
Treatments	27.9	3	9.29	...	...
Residual	35.3	21	1.68		
Total	110.4	31			

Question: What happens without blocking?

Hint: Consider $SS_{tot} = SS_{treat} + SS_{blocks} + SS_{res}$

Test of 7 different Tyres

		Cars						
		1	2	3	4	5	6	7
	1	x	x	x	x			
	2	x	x			x	x	
	3	x	x		x	x		
Tyres	4			x	x		x	x
	5			x	x	x		x
	6		x	x			x	x
	7	x				x	x	x

Blocks	Treatments			
1	1	2	3	7
2	1	2	3	6
3	1	4	5	6
4	1	3	4	5
5	2	3	5	7
6	2	4	6	7
7	4	5	6	7

- Small block size, larger number of treatments
- Non-orthogonal designs

Balanced incomplete block design

- n treatments, block size k, $(k < n)$
- Any two treatments occur together the same number of times (λ times)

First Solution: $\binom{n}{k}$ blocks, a different combination of treatments in each block.

$n = 7, k = 4 : \binom{7}{4} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$ cars

Search for smaller designs

Necessary conditions for a BIBD

b blocks, each treatment occurs r times

$$nr = bk \qquad (1)$$
$$r(k-1) = \lambda(n-1) \qquad (2)$$

(1) number of observations
(2) number of treatment pairs for a fixed treatment

Design is called symmetric if $n = b$.

Construction of BIBD

- Problem: Given k and n, how large are r,b, and λ?
- Conditions (1) and (2) are necessary but not sufficient.
- Several methods of construction exist.
- There are tables of BIBD with small sizes (Cochran & Cox 1992).
- Partially balanced block designs (PBIB) if some treatment comparisons are less important.

Analysis of BIBD

- Statistical model:

$$Y_{ij} = \mu + \beta_j + T_i + \epsilon_{ij}$$

  where T_i is the treatment effect, β_j the block effect.
- Block and treatment factor are not orthogonal, because not all combinations appear.
- Calculate first block sum of squares, then adjusted treatment sum of squares.