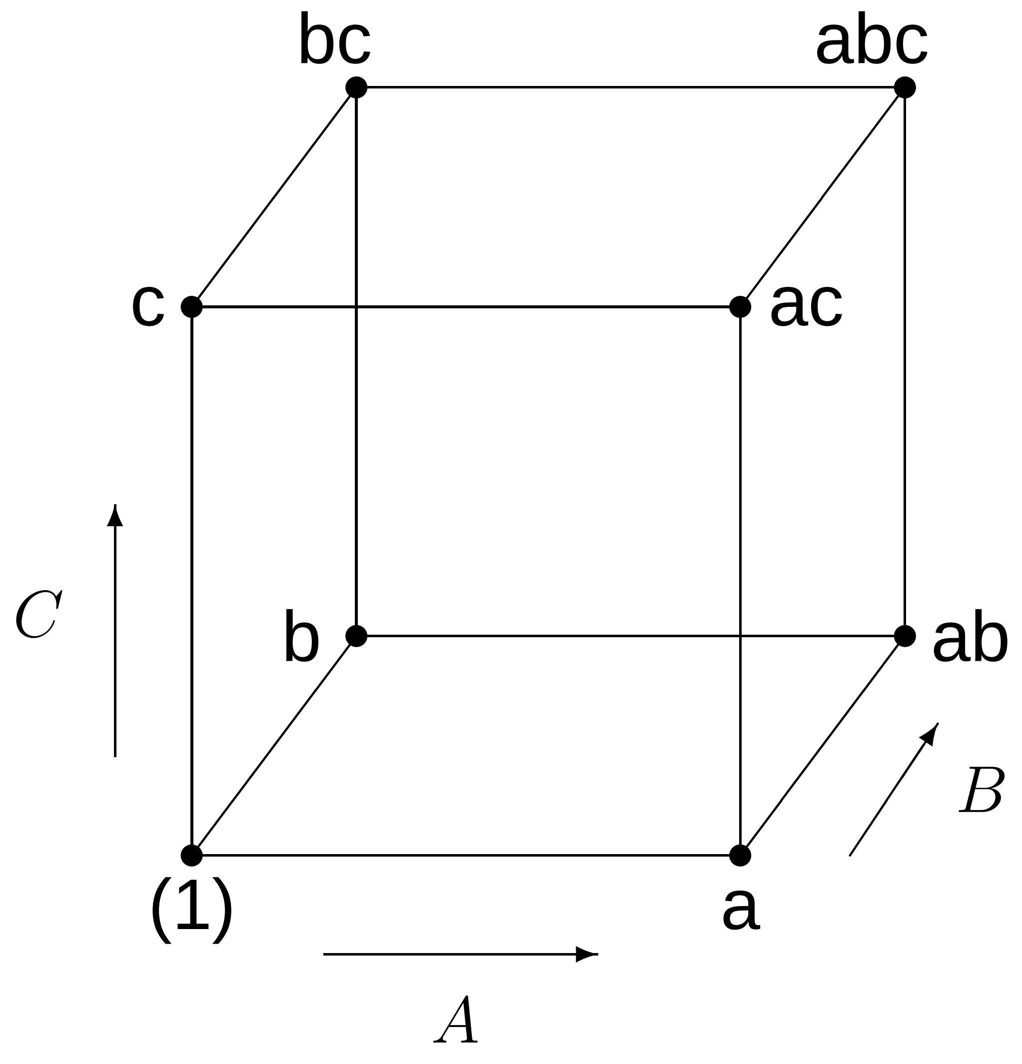
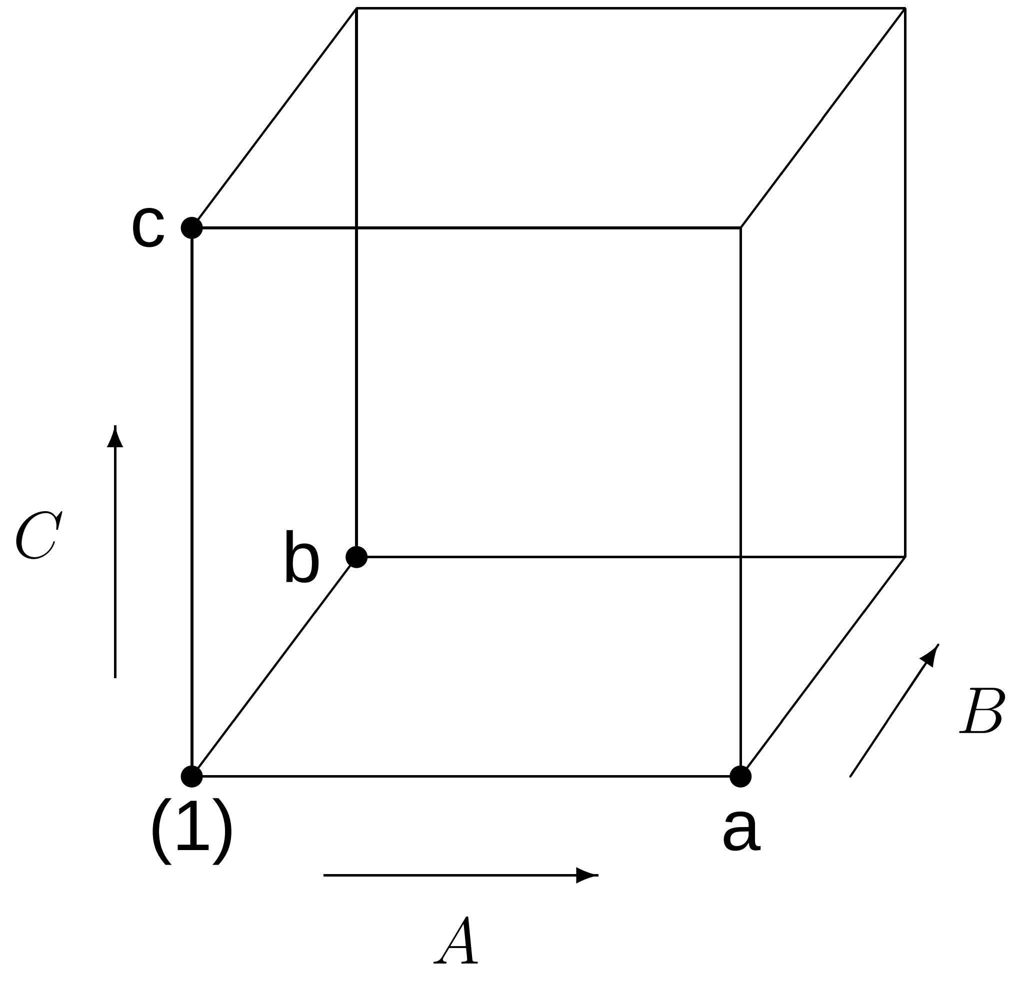
Fractional Factorials

- Too many runs for many factors
- Ignore some high-order interactions and run only a fraction of all possible runs
- How to choose the runs?

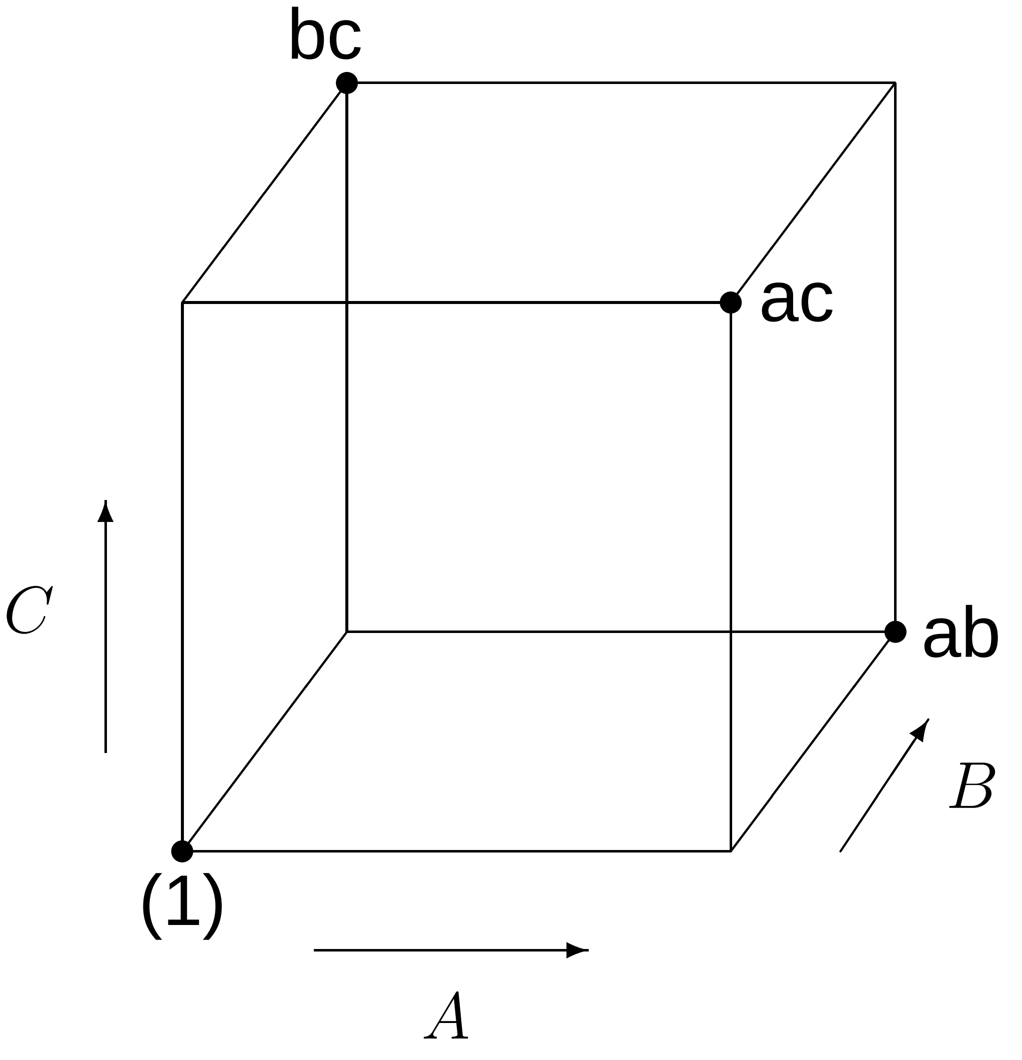
Full 2^3 factorial



Half-replicate



Optimal coverage



2^{3-1} *design*

run	A	B	C	AB	AC	BC	ABC
(1)	–	–	–	+	+	+	–
ab	+	+	–	+	–	–	–
ac	+	–	+	–	+	–	–
bc	–	+	+	–	–	+	–

$$\hat{C} = -\hat{AB}, \hat{B} = -\hat{AC}, \hat{A} = -\hat{BC}, \hat{I} = -\hat{ABC}$$

Leaf spring experiment

- An experiment to improve a heat treatment process on truck leaf springs.
- The heat treatment consists of heating in a high temperature oven, processing by a forming machine, and cooling in an oil bath.
- The response, the height of an unloaded spring, should be 8.0.
- half fraction of a 2^5 design is used to study 5 factors.

Factors and levels

Factor		Level –	Level +
A	heat temperature (°F)	1840	1880
B	heating time (seconds)	23	25
C	transfer time (seconds)	10	12
D	hold down time (seconds)	2	3
E	oil temperature (°F)	130-150	150-170

Why using fractional factorials?

- 2^5 design has 32 runs to estimate the overall mean and

Main	Interactions			
Effects	2-Factor	3-Factor	4-Factor	5-Factor
5	10	10	5	1

- 4-factor, 5-factor and even 3-factor interactions are not likely to be important. There are 10+5+1 = 16 such effects, half of the total runs!
- use a half-replicate. What price is to pay?

Design matrix

Treatment	A	B	C	D	E
(1)	–	–	–	–	–
ab	+	+	–	–	–
ac	+	–	+	–	–
bc	–	+	+	–	–
ad	+	–	–	+	–
bd	–	+	–	+	–
cd	–	–	+	+	–
abcd	+	+	+	+	–
e	–	–	–	–	+
abe	+	+	–	–	+
ace	+	–	+	–	+
bce	–	+	+	–	+
ade	+	–	–	+	+
bde	–	+	–	+	+
cde	–	–	+	+	+
abcde	+	+	+	+	+

Aliases and defining relation

- Column D is equal to the product of columns A, B and C. Estimation for main effect of D is equal to estimation for the ABC interaction: the main effect D is aliased with the interaction ABC. We write $D = ABC$.
- Then $D^2 = I = ABCD$. $I = ABCD$ is the defining relation for the 2^{5-1} design.
- 'Multiply' each side by an effect, e.g.

$$A \cdot I = A = A \cdot ABCD = A^2 \cdot BCD = I \cdot BCD = BCD$$

$$AB \cdot I = AB = AB \cdot ABCD = A^2 B^2 CD = CD$$

Aliasing structure

The complete *aliasing structure* is:

$$
\begin{aligned}
I &= ABCD & AD &= BC \\
A &= BCD & AE &= BCDE \\
B &= ACD & BE &= ACDE \\
C &= ABD & CE &= ABDE \\
D &= ABC & DE &= ABCE \\
E &= ABCDE & ABE &= CDE \\
AB &= CD & ACE &= BDE \\
AC &= BD & ADE &= BCE
\end{aligned}
$$

Construction method I

To construct a 2^{4-1} design choose one block of a 2^4 design divided into two blocks. Confound the ABCD interaction with blocks and take the principal block as half replicate.

(1)
ab
ac
bc
ad
bd
cd
abcd

2^{4-2} *Design*

Choose two confounding interactions: AB und CD.
ABCD is also confounded with blocks.

(1)
ab
cd
abcd

Aliasing structure:

$I = AB, CD, ABCD$

$A = B, ACD, BCD$

$C = ABC, D, ABD$

$AC = BC, AD, BD$

Construction method II

To construct a 2^{4-1} design start with a 2^3 design and identify the fourth factor with the ABC interaction.

Treatment	I	A	B	AB	C	AC	BC	ABC=D
(1)	+	–	–	+	–	+	+	–
a	+	+	–	–	–	–	+	+
b	+	–	+	–	–	+	–	+
ab	+	+	+	+	–	–	–	–
c	+	–	–	+	+	–	–	+
ac	+	+	–	–	+	+	–	–
bc	+	–	+	–	+	–	+	–
abc	+	+	+	+	+	+	+	+

Resolution of a design

- Resolution = length of shortest word among the $2^l - 1$ words used in the defining relations.
- In any resolution III design, main effects are not confounded with other main effects.
- In any resolution IV design, main effects are not aliased with any other main effect or 2-factor interactions.
- In any resolution V design, the main effects are not aliased with any other main effect, 2-factor or 3-factor interactions. The two-factor interactions are not aliased with any other 2-factor interaction.