Parameter estimation

- Effect Model (1):

$$Y_{ij} = \mu + A_i + \epsilon_{ij}, \qquad \sum J_i A_i = 0$$

  Estimation: $\widehat{\mu + A_i} = y_{i.} \quad \hat{\mu} = y_{..} \quad \hat{A}_i = y_{i.} - y_{..}$

  Prediction: $\hat{y}_{ij} = \hat{\mu} + \hat{A}_i = yi.,$ Residual: $r_{ij} = y_{ij} - y_{i.}$

- Effekt Modell (2):

$$Y_{ij} = \mu + A_i + \epsilon_{ij}, \qquad A_1 = 0$$

  Estimation: $\hat{\mu} = y_{1.} \quad \hat{A}_i = y_{i.} - y_{1.}$

- Mean Modell: $Y_{ij} = \mu_i + \epsilon_{ij}$ Estimation: $\hat{\mu_i} = y_{i.}$

ANOVA – Regression

- Analysis of variance models can be written as multiple regression models with indicator variables.
- Parameter estimators $y_{..}$, $y_{i.}, \ldots$ are Least Squares estimators.
- Analysis of variance models are intuitiv, treatment effects can be easily calculated and are uncorrelated.

Berliner Pfannkuchen

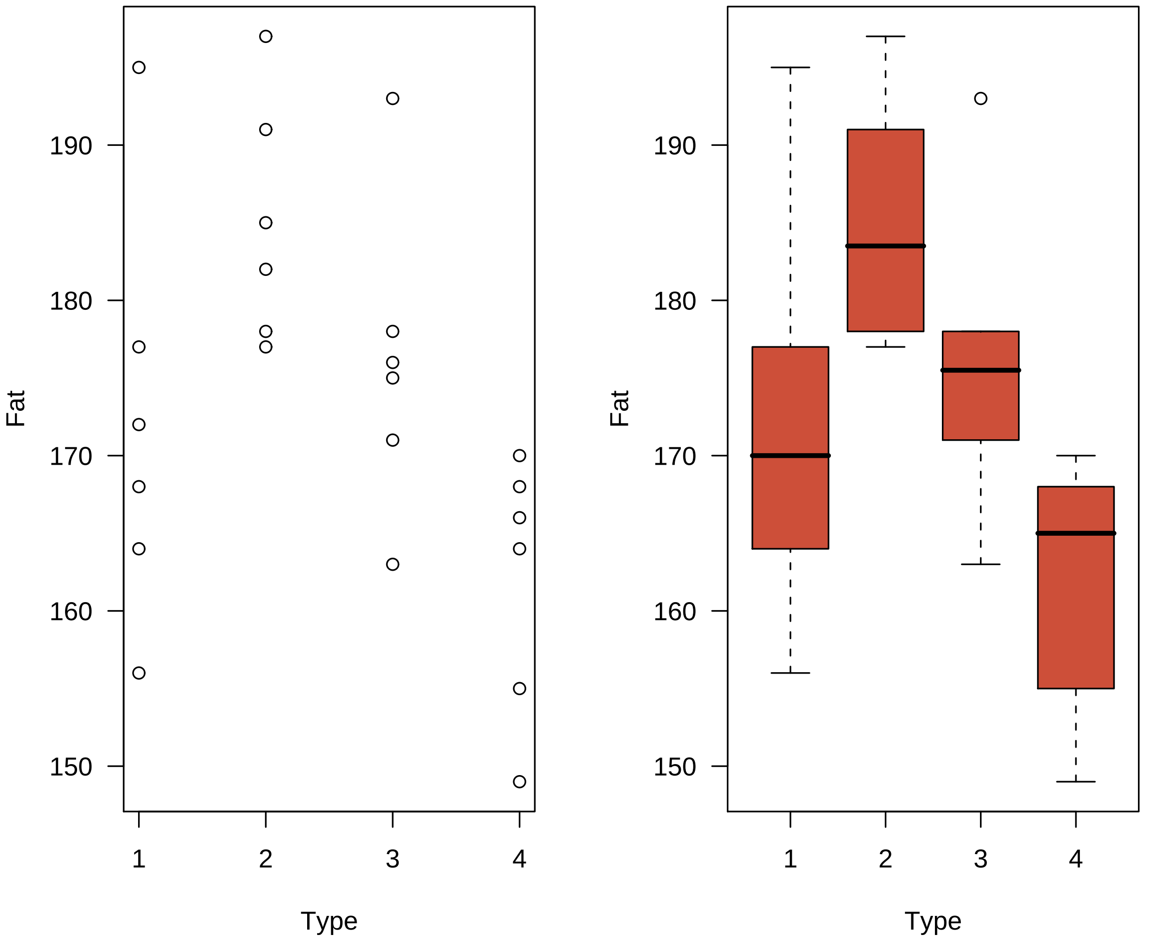
Data

Response: Fat absorption of 24 Berliner [g]

Type of Fat	Fat Absorption						Mean
1	164	172	168	177	156	195	172.0
2	178	191	197	182	185	177	185.0
3	175	193	178	171	163	176	176.0
4	155	166	149	164	170	168	162.0

balanced design: equal replication

Graphical display



R: anova table

```
> mod2=aov(fat~type,data=berliner)
> summary(mod2)
            Df Sum Sq Mean Sq F value Pr(>F)
type         3 1636.5   545.5  5.4063 0.0069**
Residuals   20 2018.0   100.9

> coef(mod2)
(Intercept)   type2       type3       type4
        172      13           4         -10
```

Design matrix

```
> model.matrix(mod2)
   (Intercept) type2 type3 type4
1            1     0     0     0
................................
6            1     0     0     0
7            1     1     0     0
................................
12           1     1     0     0
13           1     0     1     0
................................
18           1     0     1     0
20           1     0     0     1
................................
24           1     0     0     1
```

R: Multiple regression I

```
> mod2.r=lm(fat~type,data=berliner)
> summary(mod2.r)
Call:
lm(formula = fat ~ type, data = berliner)
Residuals:
       Min         1Q     Median         3Q        Max
-1.600e+01 -7.000e+00 -1.685e-14  5.250e+00  2.300e+01

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  172.000      4.101  41.943   <2e-16 ***
type2         13.000      5.799   2.242   0.0365 *
type3          4.000      5.799   0.690   0.4983
type4        -10.000      5.799  -1.724   0.1001
```

R: Multiple regression II

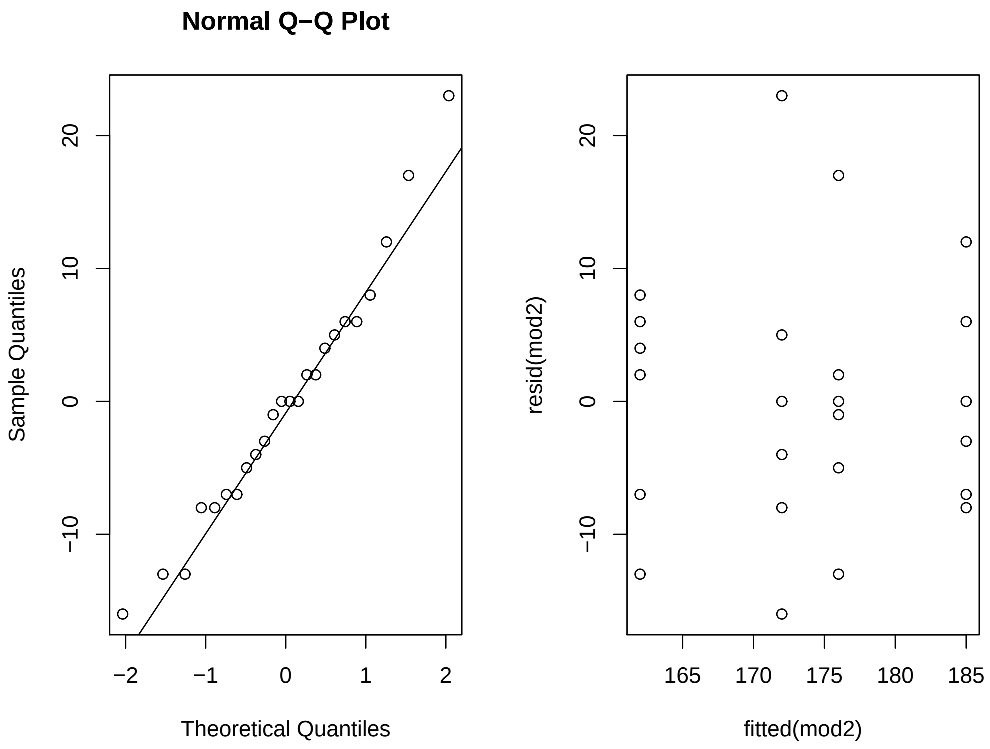
```
Residual standard error: 10.04 on 20 degrees of freedom
Multiple R-squared: 0.4478, Adjusted R-squared: 0.365
F-statistic: 5.406 on 3 and 20 DF,  p-value: 0.006876

> anova(mod2.r)
Analysis of Variance Table

Response: fat
          Df Sum Sq Mean Sq F value   Pr(>F)
type       3 1636.5   545.5  5.4063 0.006876 **
Residuals 20 2018.0   100.9
```

Model checking

Modell: $Y_{ij} = \mu + A_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim N(0, \sigma^2)$ i.i.d.

- Normal plot of residuals $r_{ij} = y_{ij} - y_{i.}$ To detect Outliers. Normal distribution not crucial in randomized experiments. Nonparametric test: Kruskal-Wallis
- Equal variances: Plot r_{ij} vs $y_{i.}$
  $\sigma^2_{min} < \frac{1}{9}\sigma^2_{max}$ (balanced designs)
  log-$\sqrt{}$ -transformation, weights
- Independent observations: Plot r_{ij} vs time, order
  more complex model, analysis

Residual plots



Treatment differences

F test significant $\Longrightarrow$ There are treatment effects.
Which? How large are the effects?

Treatment differences $y_{i.} - y_{i'.}$

Fat type 2 − Fat type 1: $185 - 172 = 13$

Fat type 3 − Fat type 1: $176 - 172 = 4$

Fat type 4 − Fat type 1: $162 - 172 = -10$

Standard error of a treatment difference:
$\sqrt{\sigma^2(1/J + 1/J)} = \sqrt{2\sigma^2/J}$, estimated by $\sqrt{2MS_{res}/J}$.

Example: $\sqrt{2 \cdot 100.9/6} = 5.799$

Are Type 2 and 1 significantly different?

t test for $H_0 : A_2 = A_1$

$$t = \frac{y_{2.} - y_{1.}}{\sqrt{2MS_{res}/J}} = \frac{13}{5.799} = 2.242 > 2.086 = t_{0.975,20}, p = 0.036$$

Confidence interval for Type 2 - Type 1:

$$13 \pm 2.086 \cdot 5.799 = 13 \pm \underbrace{12.097}_{LSD} = (0.9, 25.1)$$

Efficiency of balanced Designs

20 plots in 2 groups
10 + 10

20 plots in 2 groups
1 + 19

Standard error $y_{1\cdot} - y_{2\cdot}$

$$\hat{\sigma} \underbrace{\sqrt{\frac{1}{10} + \frac{1}{10}}}_{0.45} \qquad \hat{\sigma} \underbrace{\sqrt{1 + \frac{1}{19}}}_{1.03}$$

No big efficiency loss with moderate (2:1) imbalance.

Multiple pairwise comparisons

Are all pairs of treatments different? Is one treatment different from the others? Are there groups of similar treatments? Problem: α_E increases.

- Bonferroni correction for 6 pairwise comparisons:
  Significance level: $\alpha_T = 0.05/6$
  Critical value: $t_{1-0.05/2\cdot 6,20} = 2.927$
  Difference between Type 2 and 1 not significant.
- Tukey method for pairwise comparisons:
  critical values for the distribution of $\max|y_{i.} - y_{i'.}|$
- Dunnett's method for multiple comparisons with a control group.

Tukey method

Reject $H_0 : A_2 = A_1$, if

$$|t| > \frac{1}{\sqrt{2}} q_{1-\alpha, I, N-I}$$

with $q_{\dots}$ the quantile of the Studentized Range distribution.

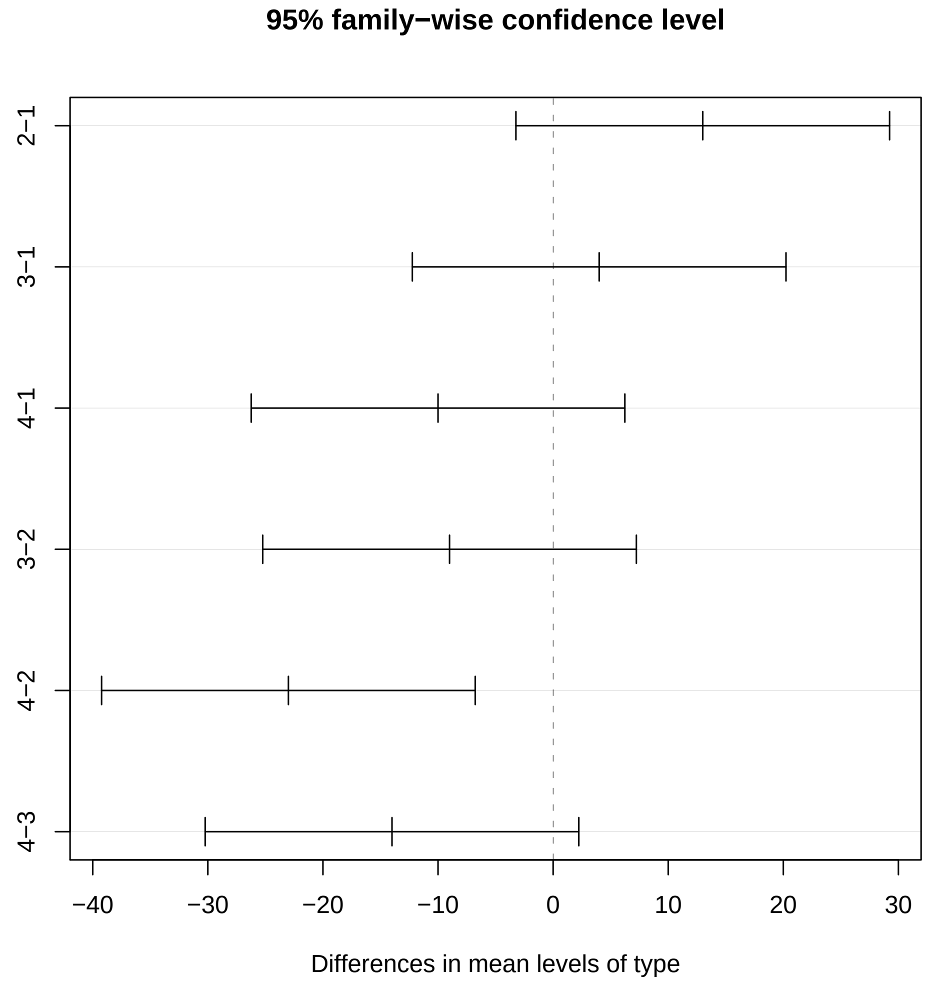
Example: $|t| > \frac{3.958}{\sqrt{2}} = 2.799$.

Type 2 and 1 do not differ significantly.

Tukey Confidence interval for Type 2 - Type 1:

$$13 \pm 2.799 \cdot 5.799 = 13 \pm \underbrace{16.23}_{HSD} = (-3.2, 29.2)$$

R: plot(TukeyHSD(mod2,“type“))



Contrasts

complex comparison: difference between fat types 1 and 4 vs 2 and 3?

Contrast:

$$C = \sum_{i=1}^{I} \lambda_i A_i \text{ with } \sum \lambda_i = 0$$

C can be estimated by

$$\begin{aligned} \hat{C} &= \sum \lambda_i \hat{A}_i = \sum \lambda_i (y_{i.} - y_{..}) \\ &= \sum \lambda_i y_{i.} - y_{..} \sum \lambda_i = \sum \lambda_i y_{i.}. \end{aligned}$$

Testing of a contrast

Reject $H_0 : \sum_{i=1}^{I} \lambda_i A_i = 0$, if

$$|t| = |\frac{\hat{C}}{\sqrt{MS_{res} \sum \frac{\lambda_i^2}{J_i}}}| > t_{0.975, N-I}$$

Equivalently,

$$F = t^2 = \frac{\hat{C}^2 / \sum \lambda_i^2 / J_i}{MS_{res}} = \frac{SS_C}{MS_{res}}$$

follows a F distribution with 1 and $N - I$ degrees of freedom. SS_C denotes the *sum of squares of the contrast* C.

Orthogonal contrasts

There are $I - 1$ linearly independent contrasts.

Two contrasts $C_1 = \sum \lambda_i A_i$ and $C_2 = \sum \lambda_i' A_i$ are called orthogonal, if $\sum \lambda_i \lambda_i' = 0$.

For balanced designs:
orthogonal contrasts $\longrightarrow$ uncorrelated estimates $\longrightarrow$ t tests nearly independent

Partitioning of Treatment Sum of Squares

$$\left(\frac{\hat{C}}{\sqrt{MS_{res}\sum\frac{\lambda_i^2}{J}}}\right)^2 = \frac{J\hat{C}^2/\sum\lambda_i^2}{MS_{res}} = \frac{SS_C}{MS_{res}} \sim F_{1,N-I}$$

SS_C= Sum of Squares of the contrast C

If $C_1, C_2, \ldots, C_{I-1}$ are orthogonal contrasts, then

$$SS_{treat} = SS_{C_1} + SS_{C_2} + \cdots + SS_{C_{I-1}}$$

Summary: Multiple Comparison

n planned , orthogonal contrasts ($n \leq I-1$)	Bonferroni (-Holm) significance level α/n
pairwise comparisons	Tukey method
comparison with a control group	Dunnett's method
complex nonorthogonal or complex unplanned comparisons	Scheffé: critical value $\sqrt{(I-1)F_{I-1,N-I,95\%}}$