

# Parameter estimation

## ■ Effect Model (1):

$$Y_{ij} = \mu + A_i + \epsilon_{ij}, \quad \sum J_i A_i = 0$$

Estimation:  $\widehat{\mu + A_i} = y_{i.}$     $\hat{\mu} = y_{..}$     $\hat{A}_i = y_{i.} - y_{..}$

Prediction:  $\hat{y}_{ij} = \hat{\mu} + \hat{A}_i = y_{i.}$ ,   Residual:  $r_{ij} = y_{ij} - y_{i.}$

## ■ Effekt Modell (2):

$$Y_{ij} = \mu + A_i + \epsilon_{ij}, \quad A_1 = 0$$

Estimation:  $\hat{\mu} = y_{1.}$     $\hat{A}_i = y_{i.} - y_{1.}$

■ Mean Modell:  $Y_{ij} = \mu_i + \epsilon_{ij}$    Estimation:  $\hat{\mu}_i = y_{i.}$

# ***Analysis of Variance – Regression***

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- Analysis of variance models can be written as multiple regression models with indicator variables.
- Parameter estimators  $y_{..}$ ,  $y_{i.}$ ,  $\dots$  are Least Squares estimators.
- Analysis of variance models are intuitiv, treatment effects can be easily calculated and are uncorrelated.

# *Berliner Pfannkuchen*



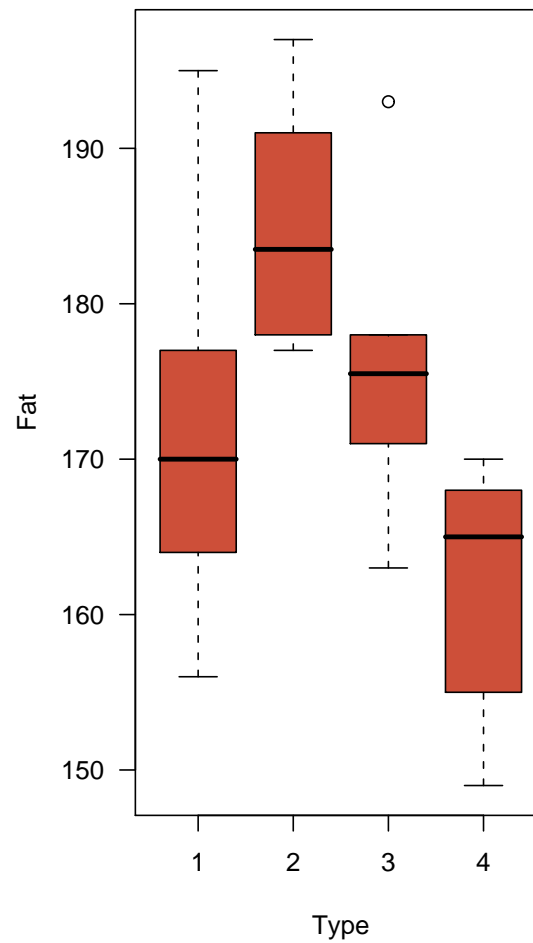
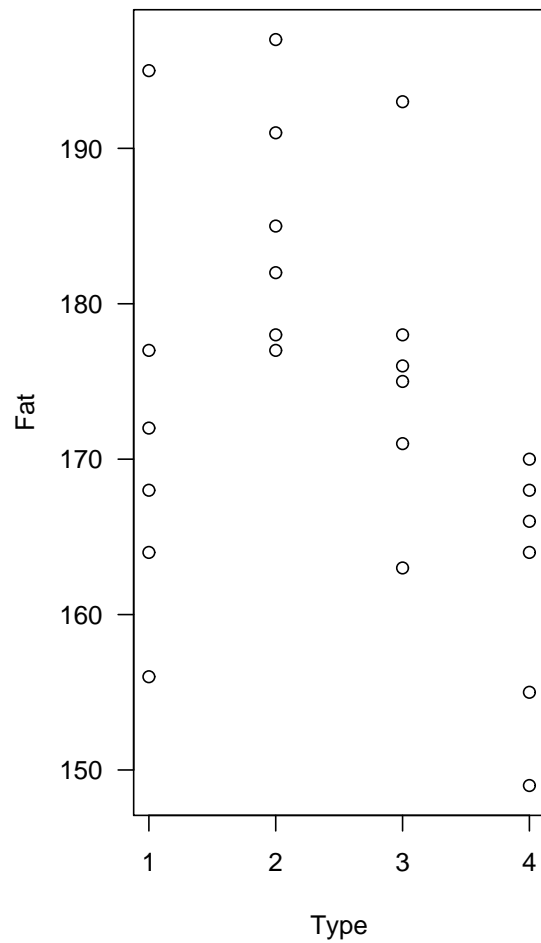
# Data

Response: Fat absorption of 24 Berliner [g]

Type of Fat	Fat Absorption						Mean
1	164	172	168	177	156	195	172.0
2	178	191	197	182	185	177	185.0
3	175	193	178	171	163	176	176.0
4	155	166	149	164	170	168	162.0

balanced design: equal replication

# Graphical display



## ***R: anova table***

```
> mod2=aov(fat~type,data=berliner)
```

```
> summary(mod2)
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
type	3	1636.5	545.5	5.4063	0.0069**
Residuals	20	2018.0	100.9		

```
> coef(mod2)
```

(Intercept)	type2	type3	type4
172	13	4	-10

# Design Matrix

```
> model.matrix(mod2)
  (Intercept) type2 type3 type4
1            1     0     0     0
.....
6            1     0     0     0
7            1     1     0     0
.....
12           1     1     0     0
13           1     0     1     0
.....
18           1     0     1     0
20           1     0     0     1
.....
24           1     0     0     1
```

# R: Multiple Regression I

```
> mod2.r=lm(fat~type,data=berliner)
```

```
> summary(mod2.r)
```

Call:

```
lm(formula = fat ~ type, data = berliner)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-1.600e+01	-7.000e+00	-1.685e-14	5.250e+00	2.300e+01

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	172.000	4.101	41.943	<2e-16	***
type2	13.000	5.799	2.242	0.0365	*
type3	4.000	5.799	0.690	0.4983	
type4	-10.000	5.799	-1.724	0.1001	



# *R: Multiple Regression II*

Residual standard error: 10.04 on 20 degrees of freedom  
Multiple R-squared: 0.4478, Adjusted R-squared: 0.365  
F-statistic: 5.406 on 3 and 20 DF, p-value: 0.006876

```
> anova(mod2.r)
```

Analysis of Variance Table

Response: fat

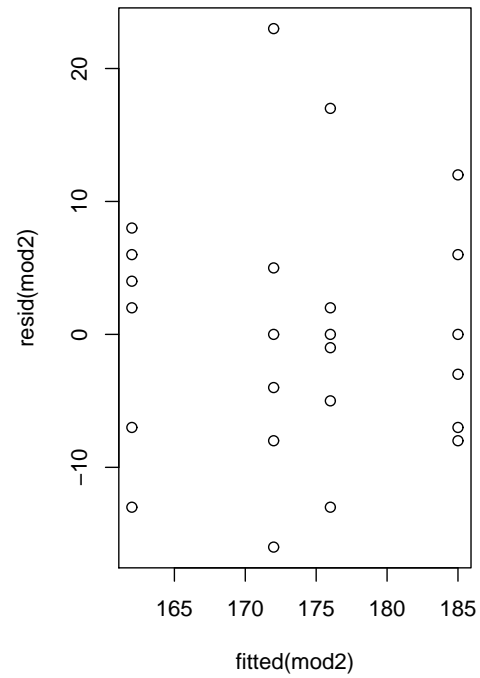
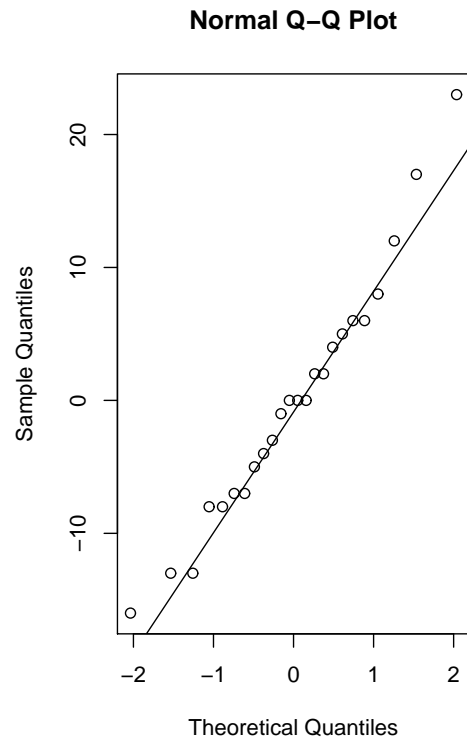
	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
type	3	1636.5	545.5	5.4063	0.006876	**
Residuals	20	2018.0	100.9			

# Model checking

Modell:  $Y_{ij} = \mu + A_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim N(0, \sigma^2)$  i.i.d.

- Normal plot of residuals  $r_{ij} = y_{ij} - y_i$ . To detect Outliers. Normal distribution not crucial in randomized experiments. Nonparametric test: Kruskal-Wallis
- Unequal variances: Plot  $r_{ij}$  vs  $y_i$ .  
 $\sigma_{min}^2 < \frac{1}{9}\sigma_{max}^2$  (balanced designs)  
log- $\sqrt{\quad}$ -transformation, weights
- independent observations: Plot  $r_{ij}$  vs time, order more complex model, analysis

# Residual plots



# Treatment differences

F test significant  $\implies$  There are treatment effects.  
Which? How large are the effects?

Treatment differences  $y_{i.} - y_{i'}$ .

$$\text{Fat type 2} - \text{Fat type 1: } 185 - 172 = 13$$

$$\text{Fat type 3} - \text{Fat type 1: } 176 - 172 = 4$$

$$\text{Fat type 4} - \text{Fat type 1: } 162 - 172 = -10$$

Standard error of a treatment difference:

$$\sqrt{\sigma^2(1/J + 1/J)} = \sqrt{2\sigma^2/J}, \text{ estimated by } \sqrt{2MS_{res}/J}.$$

$$\text{Example: } \sqrt{2 \cdot 100.9/6} = 5.799$$

# Are Type 2 and 1 significantly different?

t test for  $H_0 : A_2 = A_1$

$$t = \frac{y_{2.} - y_{1.}}{\sqrt{2MS_{res}/J}} = \frac{13}{5.799} = 2.242 > 2.086 = t_{0.975,20}, p = 0.036$$

Confidence interval for Type 2 - Type 1:

$$13 \pm 2.086 \cdot 5.799 = 13 \pm \underbrace{12.097}_{LSD} = (0.9, 25.1)$$

# Efficiency of balanced Designs

20 plots in 2 groups  
10 + 10

20 plots in 2 groups  
1 + 19

**Standard error**  $y_{1.} - y_{2.}$

$$\hat{\sigma} \underbrace{\sqrt{\frac{1}{10} + \frac{1}{10}}}_{0.45}$$

$$\hat{\sigma} \underbrace{\sqrt{1 + \frac{1}{19}}}_{1.03}$$

No big efficiency loss with moderate (2:1) imbalance.

# Multiple Comparisons

Are all pairs of treatments different? Is one treatment different from the others? Are there groups of similar treatments? Problem:  $\alpha_E$  increases.

- Bonferroni correction for 6 pairwise comparisons:  
Significance level:  $\alpha_T = 0.05/6$   
Critical value:  $t_{1-0.05/2 \cdot 6, 20} = 2.927$   
Difference between Type 2 and 1 not significant.
- Tukey method for pairwise comparisons:  
critical values for the distribution of  $\max |y_{i.} - y_{i'.}|$
- Dunnett's method for multiple comparisons with a control group.

# Tukey method

Reject  $H_0 : A_2 = A_1$ , if

$$|t| > \frac{1}{\sqrt{2}} q_{1-\alpha, I, N-I}$$

with  $q_{\dots}$  the quantile of the Studentized Range distribution.

Example:  $|t| > \frac{3.958}{\sqrt{2}} = 2.799$ .

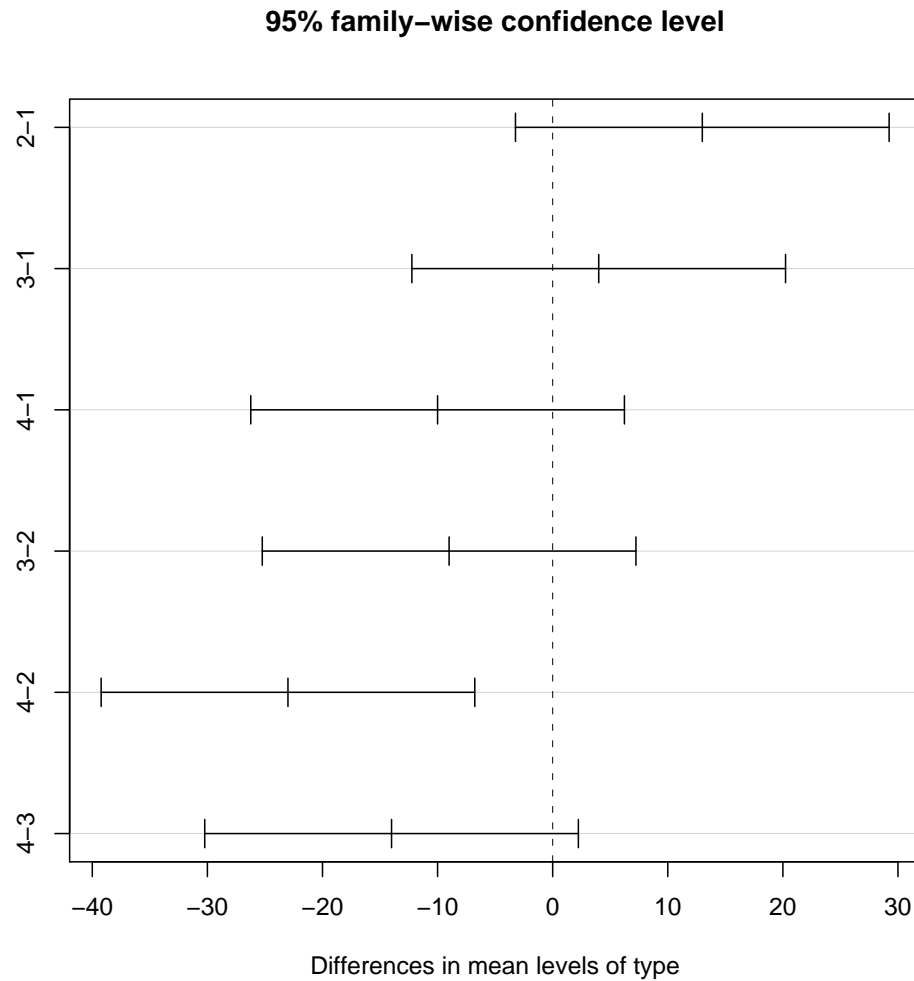
Type 2 and 1 do not differ significantly.

Tukey Confidence interval for Type 2 - Type 1:

$$13 \pm 2.799 \cdot 5.799 = 13 \pm \underbrace{16.23}_{HSD} = (-3.2, 29.2)$$



# R: `plot(TukeyHSD(mod2, "type"))`



# Definition of Contrasts

complex comparison: difference between fat types 1 and 4 vs 2 and 3?

**Contrast:**

$$C = \sum_{i=1}^I \lambda_i A_i \quad \text{mit} \quad \sum \lambda_i = 0 \quad \left( \sum J_i \lambda_i = 0 \right)$$

There are  $I - 1$  linearly independent contrasts.

Two contrasts  $C_1 = \sum \lambda_i A_i$  and  $C_2 = \sum \lambda'_i A_i$  are called **orthogonal**, if  $\sum \lambda_i \lambda'_i = 0$   $\left( \sum J_i \lambda_i \lambda'_i = 0 \right)$ .

orthogonal contrasts  $\longrightarrow$  uncorrelated estimates  $\longrightarrow$   
t tests nearly independent

# Testing of contrasts

$$\hat{C} = \sum_{i=1}^I \lambda_i \hat{A}_i = \sum_{i=1}^I \lambda_i y_{i.}, \quad \frac{\hat{C} - \sum_{i=1}^I \lambda_i A_i}{\sqrt{MSE \sum \frac{\lambda_i^2}{J}}} \sim t_{N-I}$$

Reject  $H_0 : \sum_{i=1}^I \lambda_i A_i = 0$ , if

$$\left| \frac{\hat{C}}{\sqrt{MSE \sum \frac{\lambda_i^2}{J}}} \right| > t_{0.975, N-I}$$

# Partitioning of Treatment Sum of Squares

$$\left( \frac{\hat{C}}{\sqrt{MSE \sum \frac{\lambda_i^2}{J}}} \right)^2 = \frac{J\hat{C}^2 / \sum \lambda_i^2}{MSE} = \frac{SS_C}{MSE} \sim F_{1, N-I}$$

$SS_C$  = Sum of Squares of the contrast  $C$

If  $C_1, C_2, \dots, C_{I-1}$  are orthogonal contrasts, then

$$SS_{treat} = SS_{C_1} + SS_{C_2} + \dots + SS_{C_{I-1}}$$

# Summary: Multiple Comparison

n planned , orthogonal con-  
trasts ( $n \leq I - 1$ )

Bonferroni (-Holm) signi-  
fikance level  $\alpha/n$

pairwise comparisons

Tukey method

comparison with a control  
group

Dunnett's method

complex nonorthogonal or  
complex unplanned com-  
parisons

Scheffé: critical value  
 $\sqrt{(I - 1)F_{I-1, N-I, 95\%}}$

# Power and sample size

- to detect important effects
- limitation of time, budget, subjects or plots

Power calculation:

$$\text{Power} = P_{H_A}(\text{Test sign.}) = P_{H_A}(F > F_{95\%, I-1, N-I})$$

Under  $H_A$ ,  $F$  follows a **noncentral**  $F$  distribution with centrality parameter  $\delta^2 = \frac{J \sum A_i^2}{\sigma^2}$

Given:  $I - 1, N - I, \alpha, \delta \longrightarrow$  Power  
Tables, Graphs, Software (e.g. GPower)