An example of a lower- and upper bound

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Let $Z_1, \ldots, Z_n$ be i.i.d. observations with distribution G and density g.

Recall the definition of Helliger distance $H(\cdot, \cdot)$: for two densities h_1 and h_2,

$$H^2(h_1, h_2) := \frac{1}{2} \int \left(\sqrt{h_1} - \sqrt{h_2} \right)^2 .$$

Suppose now that it is given that $g \in \mathcal{G}$. We are interested in estimating g at a fixed point z_0. The modulus of continuity for this problem is

$$m(\epsilon) := \sup\{|g(z_0) - h(z_0)| : g, h \in \mathcal{G} : \ H(g, h) \leq \epsilon\}, \ \epsilon > 0.$$

We let $\hat{g}_n(z_0)$ denote an estimator of $g(z_0)$ based on $Z_1, \ldots, Z_n$.

We also recall for this case:

Corollary 3.3.2. *Suppose that for some $r > 0$,*

$$\liminf_{\epsilon \downarrow 0} \frac{m(\epsilon)}{\epsilon^r} > 0, \ \epsilon \downarrow 0. \qquad (*)$$

Then

$$\liminf_{n \to \infty} n^{r/2} \sup_{g \in \mathcal{G}} \mathbf{E}_g |\hat{g}_n(z_0) - g(z_0)| > 0.$$

Example. Let

$$\mathcal{G} = \{g : \ [0, 1] \to [0, \infty), \ g \uparrow, \ \int g = 1\}.$$

We will show that, (*) holds with $r = 2/3$. Thus, the lower bound is of order $n^{-1/3}$.

The upper bound turns out to be of the same order, and it is achieved by the maximum likelihood estimator. To sketch the proof, we will consider a subclass

$$\mathcal{H} \subset \mathcal{G},$$

of the form

$$\mathcal{H} := \{h_\theta : \ 0 \leq \theta \leq 1/2\}.$$

We will sketch the following result: if the density g of the observations is equal to $h_0 \in \mathcal{H}$, then the maximum likelihood estimator over $\mathcal{H}$ of $g(z_0)$ converges in probability with rate $n^{-1/3}$. We will also indicate how this extends to the nonparametric maximum likelihood estimator over $\mathcal{G}$