Seminar : Inverse problems in statistics

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Chapter 6 : Consistency

1 Introduction

We want to analyse the asymptotic behaviour of an estimator, in particular with respect to the consistency of the estimator, i.e. how an estimator converges (almost surely or in probability) to the estimand for $n \to \infty$. The consistency for estimators in the context of inverse problems depends on the types of estimators. Three different approaches to prove the consistency will thus be seen here, corresponding to the three main types of estimators :

1. Plug-in estimators
2. Isotonic inverse estimators
3. Maximum likelihood estimators

2 Pointwise convergence for plug-in estimators

To check the consistency we can often use the strong law of large numbers.

3 Convergence in probability for some isotonic estimators

3.1 Two Lemmas

The Marshall's lemma gives a criterium to prove this convergence.

Lemma 3.1. *Marshall's lemma Let Φ_n be a sequence of functions on an interval I and Φ a convex function on I such that*
$\sup_{y \in I} |\Phi_n(y) - \Phi(y)| \to 0$ *as* $n \to \infty$.

Denote by Φ_n^* the convex minorant of Φ_n on I and by $\Phi_n^{*,l}$ and $\Phi_n^{*,r}$ its left and right derivative respectively. Then we have for each $x \in I^0$

$$\Phi^l(x) \leq \liminf_{n\to\infty} \Phi_n^{*,l} \leq \limsup_{n\to\infty} \Phi_n^{*,r} \leq \Phi^{*,r}(x). \tag{1}$$

This lemma requires the uniform convergence as assumption, which can be proved with the next lemma.

Lemma 3.2. *Let f_n be bounded increasing functions on an interval $I \subset \mathbb{R}$, and f be bounded, increasing and continuous on I. Suppose that $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in I$ a.s.. Then $\sup_{x\in I} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.*

3.2 Example in Wicksell's problem

The isotonic inverse estimator for V is consistent at each point x where V is continuous, provided that the distribution function F has finite first moment.

4 Maximum Likelihood Estimators

The aim is to present a way to prove consistency in the case of the maximum likelihood estimator. Let $\hat{g}_n$ be the maximizer of a concave loglikelihood function l over a convex class of densities $\mathcal{G}_n \subset \mathcal{G}$. Than for any $g_n \in \mathcal{G}_n$ and for each $\varepsilon \in (0,1)$ we have

$$l(\hat{g}_n + \varepsilon(g_n - \hat{g}_n)) - l(\hat{g}_n) \leq 0$$

Hence we get

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} l(\hat{g}_n + \varepsilon(g_n - \hat{g}_n)) - l(\hat{g}_n) \leq 0 \tag{2}$$

If the underlying density is an element of $\mathcal{G}_n$ it is possible to prove consistency of the MLE. More precisely in the following situation:
Suppose that $Z_1, Z_2, ...$ are i.i.d. random variables with a convex and decreasing density g_0 on the interval $[0, \infty)$. As before we denote with $\hat{g}_n$ the maximizer of the loglikelihood function

$$l(g) = \int_0^\infty \log g(z) dG_n(z)$$

over the class $\mathcal{G}_n$ of decreasing piecewise linear convex densities with all changes of slope only for the set of observations. G_n is the empirical distribution function. With the use of equation (2) it can be shown that $(\hat{g}_n)$ convergence uniformly to g_0 on intervals of the form $[c, \infty)$ for $c > 0$, i.e.

$$\sup_{c\leq z<\infty} \mid \hat{g}_n(z) - g_0(z) \mid \to 0 \text{ almost surely.}$$