Counting Processes and Martingal I: Part 2

Claudia Soldini

08.05.2006

1 The Doob-Meyer Decomposition

For a submartingal X it is often possible to find an increasing process A such that $X - A$ is a martingal.
Using additional restrictions on X and A, A is unique. The unique decomposition of $X = M + A$ is called the *Doob-Meyer decomposition*.
Now we examine the conditions on A which are sufficient for the existence and uniqueness of the Doob-Meyer decomposition. The central condition is the *predictability*.

Definition 1 *Let* $(\Omega,\ P,\ \mathrm{F})$ *be a probability space with filtration* $\{\mathrm{F}_t\ ;\ t \geq 0\}$. *The* σ*-algebra on* $[0, \infty) \times \Omega$ *generated by all sets of the form*

$$[0] \times A, A \in \mathrm{F}_0,$$

and

$$(a, b] \times A, 0 \leq a < b < \infty, A \in \mathrm{F}_a,$$

is calles the predictable σ-algebra *for the filtration* F_t.

Definitions 2 *A process X is called* predictable *with respect to a filtration if, as a mapping from* $[0, \infty) \times \Omega$ *to* $\mathbb{R}$*, it is measurable with respect to the predictable* σ*-algebra generated by the filtration. We call X an* F_t-predictable process.

Preposition 1 *Let X be an* F_t*-predictable process. Then* $\forall t > 0$*, X(t) is* F_{t-}*-measurable.*

The main use of predictability of a process Q is its F_{t-}-measurability, implying

$$E\{Q(t)|F_{t-}\} = Q(t) a.s.$$

Now we have to introduce an important integrability condition which is relied in the version of the Doob-Meyer decomposition we use.

Definition 3 *A collection of random variables $\{X_t : t \in \tau\}$, where τ is an arbitrary index set, is* uniformly integrable *if*

$$\lim_{n\to\infty} \sup_{t\in\tau} E(|X_t| 1_{\{|X_t|>n\}}) = 0$$

The following proposition gives conditions often used to check uniform integrability.

Preposition 2 *The collection $\{X_t : t \in \tau\}$ is uniformly integrable if and only if the following two conditions are satisfied:*

1. $\sup_{t\in\tau} E|X_t| < \infty$
2. *For every $\epsilon > 0$, there exist $\delta(\epsilon)$ such that for any set A with $P(A) < \delta$,*

$$\sup_{t\in\tau} \int_A |X_t| dP < \epsilon$$

Now we are ready to understand a first version of the Doob-Meyer decomposition. The version given here is not the most general, since we state it only for the case of right-continuous, nonnegative submartingales. For right-continuous submartingales of arbitrary sign, either the submartingal must satisfy additional regularity condition or the process M will have somewhat less structure than a martingale (i. e. it will be a local martingale, as we will see later).

Theorem(Doob-Meyer Decomposition) 1 *Let X be a right-continuous nonnegative submartingale with respect to a stochastic basis $(\Omega, \mathrm{F}, \{\mathrm{F}_t; t \geq 0\}, P)$. Then there exists a right-continuous martingale M and an increasing right-continuous predictable process A such that $EA(t) < \infty$ and*

$$X(t) = M(t) + A(t) a.s.$$

for any $t \geq 0$. If A(0)=0 a. s., and if X=M'+A' is another such decomposition with A'(0)=0, then for any $t \geq 0$,

$$P\{M'(t) \neq M(t)\} = 0 = P\{A'(t) \neq A(t)\}.$$

If in addition X is bounded, then M is uniformly integrable and A is integrable.

The decomposition theorem states that for any right-continuous non-negative submartingale X there is a unique increasing right-continuous predictable process A such that A(0)=0 and $X - A$ is a martingale. Since any adapted nonnegative increasing process with finite expectation is a submartingale, there is a unique process A so that for any counting process N with finite expectation, $N - A$ is a martingale. We summarize this in the next corollary.

Corollary 1 *Let $\{N(t) : t \geq 0\}$ be a counting process adapted to a right-continuous filtration $\{\mathrm{F}_t; t \geq 0\}$ with $EN(t) < \infty$ for any t. Then there exists a unique increasing right-continuous F_t-predictable process A such that*

- *A(0)=0 a. s.*
- *$EA(t) < \infty$ for any t,*
- *$\{M(t) = N(t) - A(t) : t \geq 0\}$ is a right-continuous F_t-martingale*

Definition 4 *The process A in the Doob-Meyer decomposition is called the* compensator *for the submartingale X.*

In theorem 2.2 (part 1) we had, under a sufficient and necessary condition that

$$M(t) = N(t) - \int_0^t 1_{\{X \geq u\}} \lambda(u) du$$

is an F_t-martingale. That theorem showed that the integrated conditional hazard rate is the compensator process for the simple counting process denoting the time of an observed failure time subject to censoring.

2 Local Martingales

2.1 Introduction

In this chapter we will work with more general setting then in chapter 1. First we will explore the idea of localization and its use in extending the Doob-Meyer decomposition and then we will re-establish the results from previous chapter for local martingales of the form $N - A$.

2.2 Localization of stochastic processes and the Doob-Meyer Decomposition

Definition 1 Let $\{\mathrm{F}_t : t \geq 0\}$ be a filtration on a probability space. A nonnegative random variable τ is a *stopping time* with respect to $\{F_t\}$ if $\{\tau \leq t\} \in F_t$ for all $t \geq 0$.

If τ is thought as the time an event occurs, then τ will be a stopping time if the information in F_t specifies whether or not the event has happened by time t.

Definition 2 An increasing sequence of random times τ_n, n=1,2,... is called *a localizing sequence with respect to a filtration* if the following hold true

1. Each τ_n is a stopping time relative relative to the filtration

2. $\lim_{n\to\infty} \tau_n = \infty$ *a. s.*

A property is said to hold *locally* for a stochastic process if the property is satisfied by the stopped process $X_n = \{X(t \wedge \tau_n) : t \geq 0\}$ for each n, where τ_n form a localizing sequence.

Definition 3

1. A stochastic process $M = \{M(t) : t \geq 0\}$ is a *local martingal (submartingale) with respect to the filtration* $\{\mathrm{F}_t : t \geq 0\}$ if there exists a localizing sequence $\{\tau_n\}$ such that, for each n, $M_n = \{M(t \wedge \tau_n) : 0 \leq t < \infty\}$ is an F_t-martingale (submartingale).

2. If M_n above is a martingale and a square integrable process, M_n is called a *square integrable martingale* and M is called a *local square integrable martingale.*

3. An adapted process $X = \{X(t) : t \geq 0\}$ is called *locally bounded* if, for a suitable localizing sequence $\{\tau_n\}$, $X_n = \{X(t \wedge \tau_n) : t \geq 0\}$ is a bounded process for each n.

Lemma 1 Any martingale is a local martingal

Lemma 2 An F_t -local martingal M is a martingal, if for any fixed t, $\{M(t \wedge \tau_n) : n = 1, 2, \ldots\}$ is a uniformly integrable sequence, where $\{\tau_n\}$ is a localizing sequence for M.

We arrive now to the Optional Stopping Theorem, which says that stopping a martingale or a submartingale at a random time does not disturb its special structure.

Theorem(Optional Stopping Theorem) 1 *Let $\{X(t) : 0 \leq t < \infty\}$ be a right-continuous* F_t*-martingale (respectively, submartingale) and let τ be an* F_t*-stopping time. Then $\{X(t \wedge \tau) : 0 \leq t < \infty\}$ is a martingale (respectively, submartingale).*

The next result shows that there is some flexibility in choosing localizing sequences

Lemma 3 *Suppose M is a right-continuous local square integrable martingale on $[0,\infty)$, with localizing sequence $\{\tau_n{}^*\}$. Let $\{\tau_n\}$ be another increasing sequence of stopping times with $\tau_n \to \infty$ and $\tau_n \leq \tau_n{}^*$ a. s. Then $\{M(t) : t \geq 0\}$ also is right-continuous local square integrable martingale on $[0,\infty)$, with localizing sequence $\{\tau_n\}$.*

We can now state the version of the Doob-Meyer Decomposition for nonnegative local submartingales.

Theorem (Extended Doob-Meyer Decomposition) 2 *Let $\{X(t) : t \geq 0\}$ be a nonnegative right-continuous F_t-local submartingale with localizing sequence $\{\tau_n\}$, where $\{\mathrm{F}_t : t \geq 0\}$ is a right-continuous filtration. Then there exists a unique increasing right-continuous predictable process A such that $A(0) = 0$ a. s., $P(A(t) < \infty) = 1$ for alle $t > 0$, and $X - A$ is a right-continuous local martingale. At each t, A(t) may be taken as the a.s. $\lim_{n\to\infty} A_n(t)$, where A_n is the compensator of the stopped submartingale $X(\cdot \wedge \tau_n)$.*

2.3 The martingale $N - A$ revisited

The extended Doob-Meyer Decomposition implies the existence of a compensator A for any counting process N so that $N - A$ is a local martingale. With other words it can be used to represent an arbitrary counting process as the sum of a local martingale and a predictable increasing process.

Theorem 1 *Let N be an arbitrary counting process, then there exists a unique right-continuous predictable process A such that $A(0) = 0$ a. s., $A(t) < \infty$ a.s. for any t, and the process $M = N - A$ is local martingales*

Consistent with earlier terminology, the process A in the previous decomposition of an arbitrary counting process will be called a *compensator*. Let's look at its characterization.

Theorem 2 *Let N be a counting process and let A be its unique compensator in the Extended Doob-Meyer Decomposition Theorem. Then:*

1. *A is a locally bounded process, and*
2. *$\triangle A(t) \equiv A(t) - \lim_{s\to t} A(s) \leq 1$ a.s. for all $t \geq 0$.*

The next result provides a method for determining $EN(t)$ and establishes a condition under which $M = N - A$ is a martingale.

Lemma 4 *Suppose N is a counting process. Then $EN(t) = ENA(t)$ for any t, where A is the compensator for N. If $EA(t) < \infty$ for all t, then $M = N - A$ is a martingale.*