COUNTING PROCESSES AND MARTINGALES I (PART 1)

Morosoli Giovanni

1 Stochastic processes

Definition 1.1 (random variable)
Let $(\Omega, \mathscr{F}, P)$ be a probability space.
A function Z from Ω to $\mathbb{R}$ is called a random variable (or measurable relative to $\mathscr{F}$) if $\{Z \leq x\} = \{\omega \in \Omega : Z(\omega) \leq x\} \in \mathscr{F}, \forall x$.

Definition 1.2 (stochastic process; trajectory)
A stochastic process is a family of random variables $X = \{X_t : t \in \Gamma\}$, where Γ is the so called index set.
For a stochastic process X, the function $X(., \omega) : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $\omega \in \Omega$, are called sample paths or trajectories of X.

Definition 1.3 (equivalent random variables)
Two random variables X and Y are called equivalent if $P\{X \neq Y\} = 0$ i.e. $X = Y$ almost surely.

Definition 1.4 (modification)
A stochastic process X is called a modification of another process Y, if for any t, X_t and Y_t are equivalent random variables.

Proposition 1.5
If the process X is a modification of the process Y, then X and Y have the same finite-dimensional distributions.

Definition 1.6 (indistinguishable)
The process X and Y are indistinguishable if $P\{\omega : X_t = Y_t \ \forall t \geq 0\} = 1$

Proposition 1.7
If the processes X and Y are indistinguishable, then the sample paths of X coincide with those of Y for almost all ω.

Proposition 1.8
If both the processes X and Y are left-continuous or both right-continuous, then: X and Y are indistinguishable $\Longleftrightarrow$ X and Y are modifications of one another.

Definition 1.9 (integrable; square integrable; bounded)
A stochasstic process X is

- *Integrable if $sup_{0\leq t<\infty} E\left[|X_t|\right] < \infty$;*
- *Square integrable if $sup_{0\leq t<\infty} E\left[(X_t)^2\right] < \infty$*
- *Bounded if there exists a finite constant C such that:*
  $P\left\{sup_{0\leq t<\infty}|X_t| < C\right\} = 1$

The following definitions will allow a mathematical formualtion of the concept of „information accruing over time".

Definition 1.10 (filtration; stochastic basis)

- *A family of sub-$\sigma-$algebras $\{\mathscr{F}_t \ : \ t \geq 0\}$ of a $\sigma-$algebra $\mathscr{F}$ is called increasing if $s \leq t \Longrightarrow \mathscr{F}_s \subset \mathscr{F}_t$ $(i.e.: for\ s \leq t, A \in \mathscr{F}_s \Rightarrow A \in \mathscr{F}_t)$. An increasing family of sub $\sigma-$algebras is called a filtration.*
- *When $\{\mathscr{F}_t \ : \ t \geq 0\}$ is a filtration, the $\sigma-$algebra $\bigcap_{h>0} \mathscr{F}_{t+h}$ is usually denoted by $\mathscr{F}_{t+}$. The corresponding limit from the left, $\mathscr{F}_{t-}$, is the smallest $\sigma-$algebra containig all the sets in $\bigcup_{h>0} \mathscr{F}_{t-h}$*
- *A filtration $\{\mathscr{F}_t \ : \ t \geq 0\}$ is right-continuous if, for any t $\mathscr{F}_{t+} = \mathscr{F}_t$.*
- *A stochastic basis is a probability space $(\Omega, \mathscr{F}, P)$ equipped with a right-continuous filtration $\{\mathscr{F}_t \ : \ t \geq 0\}$, and is denoted by:*
  $(\Omega, \mathscr{F}, \{\mathscr{F}_t \ : \ t \geq 0\}, P)$.
- *A stochastic basis is complete if $\mathscr{F}$ contains any subset of a P-null set and if each $\mathscr{F}_t$ contains all P-null sets of $\mathscr{F}$.*

Natural filtrations are families with $\mathscr{F}_t = \sigma\{X_s \ : \ 0 \leq s \leq t\}$ and are called histories of the stochastic process.
In this case, $\mathscr{F}_t$ „contains the information" generated by the process X on [0, t].

Definition 1.11 (adapted)
A stochastic process $X = \{X_t \ : \ t \geq 0\}$ *is adapted to a filtration if, for every* $t \geq 0$, X_t *is* $\mathscr{F}_t$*-measurable.*

Remark 1.12
Any process is adapted to its history.

Definition 1.13 (counting process)
A counting process is a stochastic process $\{N_t, t \geq 0\}$ *adapted to a filtration* $\{\mathscr{F}_t, t \geq 0\}$ *with* $N_0 = 0$ *and* $N_t < \infty$ *a.s., and whose paths are with probability one right-continuous, piecewise constant, and have only jump discontinuities, with jumps of size +1.*

Definition 1.14 (conditional expectation)
Let Y *be a random variable on a probability space* $(X, \mathscr{F}, P)$*, and let* $\mathscr{G}$ *be a sub* $\sigma-$*algebra of* $\mathscr{F}$*.We define* X *as a random variable satisfying:*

- X *is* $\mathscr{G}-$*measurable; and*
- $\int_B Y dP = \int_B X dP, \forall B \in \mathscr{G}.$

The variable X *is called the conditional expectation of* Y *given* $\mathscr{G}$*, and is denoted by* $E\left(Y|\mathscr{G}\right)$*.*

Proposition 1.15 (Properties of conditional expectation)
Let $(\Omega, \mathscr{F}, P)$ *be a probability space,* X*,* Y *random variables on this space and* $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$ *sub* $\sigma-$*algebras. Then:*

1. $\mathscr{F}_t = \{\varnothing, \Omega\}\,,\ E\left[X|\mathscr{F}_t\right] = E\left[X\right]$ *a.s.*
2. $E\left[E\left[X|\mathscr{F}_t\right]\right] = E\left[X\right]$
3. *If* $\mathscr{F}_s \subset \mathscr{F}_t$*, then:*
   $E\left[E\left[X|\mathscr{F}_s\right]|\mathscr{F}_t\right] = E\left[E\left[X|\mathscr{F}_t\right]|\mathscr{F}_s\right] = E\left[X|\mathscr{F}_s\right]$ *a.s.*
4. *If* $\sigma(Y) \subset \mathscr{F}_t$*, then* $E\left[XY|\mathscr{F}_t\right] = Y E\left[X|\mathscr{F}_t\right]$ *a.s.*
5. *Let* X*,* Y *be independent random variables, and* $\mathscr{G} = \sigma(X)$*, then:*$E\left[Y|\mathscr{G}\right] = E\left[Y\right]$
6. $E\left[aX + bY|\mathscr{F}_t\right] = aE\left[X|\mathscr{F}_t\right] + bE\left[Y|\mathscr{F}_t\right]$ *a.s,* a*,* b *constants.*
7. *(Jensen inequality) Let* g *be a real convex function, then:* $E\left[g(X)|\mathscr{F}_t\right] \geq g\left(E\left[X|\mathscr{F}_t\right]\right)$*.*

Notation: For X, Y two tandom variables: $E[Y|X] := E[Y|\sigma(X)]$
Generally: If $\mathscr{A}$ *is a family of random variables, then* $E[Y|\mathscr{A}] := E[Y|\sigma(\mathscr{A})]$

Definition 1.16 (martingale)
Let $X = \{X_t : t \geq 0\}$ *be a right-continuous stochastic process with left-hand limits and* $\{\mathscr{F}_t : t \geq 0\}$ *a filtration, defined on a common probability space. Then* X *is called a martingale with respect to* $\{\mathscr{F}_t : t \geq 0\}$ *if:*

1. X *is adapted to* $\{\mathscr{F}_t : t \geq 0\}$,
2. $E[|X_t|] < \infty, \forall t < \infty$,
3. $E[X_{t+s}|\mathscr{F}_t] = X_t$ a.s. $\forall s \geq 0, t \geq 0$.

X *is called a submartingale if property 3 is replaced by* $E[X_{t+s}|\mathscr{F}_t] \geq X_t$ a.s.*, and a supermartingale if it is replaced by* $E[X_{t+s}|\mathscr{F}_t] \leq X_t$ a.s.

Proposition 1.17
Let X *be a martingale with respect to a filtration* $\{\mathscr{F}_t : t \geq 0\}$.
Then $E[X_t|\mathscr{F}_{t-}] = X_{t-}$ a.s.

2 The martingale M = N - A

Proposition 2.1
Let

- *T and U be nonnegative, independent random variables*
- *$X = min\ (U,\ T)$*
- *$\{N_t\ :\ t \geq 0\}$ be a counting process given at time t by* $N_t = I_{\{X \leq t, \delta=1\}} = \delta I_{\{T \leq t\}}$, *where* $\delta = I_{\{T \leq U\}}$
- $\lambda(t) = lim_{\Delta t \downarrow 0}(\Delta t)^{-1} P\left[t \leq T < t + \Delta t\right]\left[P\left[T \geq t\right]\right]^{-1}$
- *$\{A_t\ :\ t \geq 0\}$ be a stochastic process given by* $A_t = \int_0^t I_{\{X \geq u\}}\lambda(u)du$

Then the process $\{M_t\ :\ t \geq 0\}$ given at time t by:
$M_t = I_{\{X \leq t, \delta=1\}} - \int_0^t I_{\{X \geq u\}}\lambda(u)du = N_t - A_t$ *is a martingale.*

Theorem 2.2
Let T be an absolutely continous random variable and U a random variable with an arbitrary distribution. Set $X = min(T,\ U)$, $\delta = I_{\{T \leq U\}}$, and let δ denote the hazard function for T. Define $N_t = I_{\{X \leq t, \delta=1\}}$, $N_t^U = I_{\{X \leq t, \delta=0\}}$ and $\mathscr{F}_t = \sigma\left\{N_u, N_u^U\ :\ 0 \leq u \leq t\right\}$.
Then the process M given by $M_t = N_t - \int_0^t I_{\{X \geq u\}}\lambda(u)du$ is an $\mathscr{F}_t-$martingale if and only if $\lambda(t) = \frac{-\frac{\partial}{\partial u}P[T \geq u, U \geq t]|_{u=t}}{P[T \geq t, U \geq t]}$ whenever $P\left[X > t\right] > 0$.

Remark 2.3
The hazard function $\lambda(t)$ is also called „net hazard function", and $\lambda'(t) = \frac{-\frac{\partial}{\partial u}P[T \geq u, U \geq t]|_{u=t}}{P[T \geq t, U \geq t]}$ is called „crude hazard function".

Definition 2.4 (cumulative hazard function)
Let T be a random variable with arbitrary distribution function $F(t) = P\left[T \leq t\right]$. The cumulative hazard function Λ for T is given by $\Lambda(t) = \int_0^t \frac{dF(u)}{1-F(u-)}$.

Theorem 2.5
Let T and U be two random variables and let $X = min\ (T,\ U)$, $\delta = I_{\{T \leq U\}}$, $N_t = I_{\{X \leq t, \delta=1\}}$, $N_t^U = I_{\{X \leq t, \delta=0\}}$ and $\mathscr{F}_t = \sigma\left\{N_u, N_u^U\ :\ 0 \leq u \leq t\right\}$.
Then the process M given by $M_t = N_t - \int_0^t I_{\{X \geq u\}}d\Lambda(u)$ is an $\mathscr{F}_t-$martingale if and only if $\frac{dF(z)}{1-F(z-)} = \frac{-dP[T \geq z; U \geq T]}{P[T \geq z; U \geq z]}$ for all z such that $P\left[T \geq z, U \geq z\right] > 0$.