Representation of an ARMA(p,q) as state space model

The result

Let (Y_t) be a stationary ARMA(p,q)-model. We set $k = \max(p, q+1)$. By the definition of ARMA-models

$$Y_t = \sum_{j=1}^{k} \phi_j Y_{t-j} + \sum_{j=0}^{k-1} \theta_j \varepsilon_{t-j} \tag{1}$$

where $\theta_0 = 1$, $\theta_j = 0$ for $j > q$, $\phi_j = 0$ for $j > p$ and the innovations ε_t are i.i.d. with $E[\varepsilon_t] = 0$.

We define the state X_t to be $(Y_t, Y_{t+1|t}, \ldots Y_{t+k-1|t})^T$ where

$$Y_{s|t} = E[Y_s \mid Y_t, Y_{t-1}, \ldots]. \tag{2}$$

We claim that under the usual conditions on the zeroes of the polynomials associated with the AR and MA coefficients, (X_t, Y_t) is a linear state space model.

Proof

The observation equation $Y_t = (1, 0, \ldots, 0)X_t$ is obvious. We have to show that

$$X_{t+1} = F X_t + U_{t+1} \tag{3}$$

for a suitable matrix F and a noise vector U_{t+1}.

The key property is

$$E[\varepsilon_s \mid Y_t, Y_{t-1}, \ldots] = E[\varepsilon_s] = 0 \quad (s > t), \tag{4}$$

$$E[\varepsilon_s \mid Y_t, Y_{t-1}, \ldots] = \varepsilon_s \quad (s \leq t). \tag{5}$$

If you are familiar with the concepts of causality and invertibility, (5) follows immediately from invertibility and (4) from causality. Otherwise, use the conditions on the zeroes of the polynomials associated with the AR and MA coefficients, to show that

$$Y_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j} \tag{6}$$

for some coefficients α_j and

$$\varepsilon_t = \sum_{j=0}^{\infty} \beta_j Y_{t-j}. \tag{7}$$

for some other coefficients β_j. This means that knowing $(Y_t, Y_{t-1}, \ldots)$ is equivalent to knowing $(\varepsilon_t, \varepsilon_{t-1}, \ldots)$. From this, (5) is obvious and (4) follows because the ε_t's are independent.

Next we show how (4) and (5) imply the theorem. By the linearity of the conditional expectation, it follows from (1) that

$$Y_{t+i|t} = \sum_{j=1}^{i-1} \phi_j Y_{t+i-j|t} + \sum_{j=i}^{k} \phi_j Y_{t+i-j} + \sum_{j=i}^{k-1} \theta_j \varepsilon_{t+i-j} \tag{8}$$

for $i \geq 1$ (empty sums are equal to zero). In particular,

$$Y_{t+1|t} = Y_{t+1} - \varepsilon_{t+1}, \tag{9}$$

and

$$Y_{t+k|t} = \sum_{j=1}^{k-1} \phi_j Y_{t+k-j|t} + \phi_k Y_t. \tag{10}$$

Hence $Y_{t+k|t}$ is a linear combination of the components of the state vector X_t. By induction, the same is true for any $i \geq k$. In other words, X_t contains all the information from the past needed to predict all future values, in accordance with the intuitive meaning of the state vector.

In order to prove (3), we write equation (8) with t replaced by $t+1$ and i replaced by $i-1$

$$Y_{t+i|t+1} = \sum_{j=1}^{i-2} \phi_j Y_{t+i-j|t+1} + \sum_{j=i-1}^{k} \phi_j Y_{t+i-j} + \sum_{j=i-1}^{k-1} \theta_j \varepsilon_{t+i-j}. \tag{11}$$

Taking the difference between (11) and (8), we obtain for $2 \leq i \leq k$

$$Y_{t+i|t+1} = Y_{t+i|t} + \phi_{i-1}(Y_{t+1} - Y_{t+1|t}) + \sum_{j=1}^{i-2} \phi_j (Y_{t+i-j|t+1} - Y_{t+i-j|t}) + \theta_{i-1}\varepsilon_{t+1}. \tag{12}$$

Hence by induction for $2 \leq i \leq k$

$$Y_{t+i|t+1} = Y_{t+i|t} + g_i \; \varepsilon_{t+1}, \quad g_i = \sum_{j=1}^{i-1} \phi_j g_{i-j} + \theta_{i-1}. \tag{13}$$

Equation (9) defines the first row of F, and equation (13) defines rows $2, \ldots, k-1$ of F. The last row of F is obtained by using (10) in addition. Moreover, the noise U_{t+1} is equal to $\varepsilon_{t+1} \cdot (1, g_2, \ldots, g_k)^T$.