Skeleton of AR(p) models

without innovations

$$x_t = \sum_{j=1}^{p} \phi_j x_{t-j}, \quad \phi_p \neq 0. \qquad (1)$$

Theorem 1. *All solutions of (1) build a vector space of dimension p. The basis vectors correspond to the roots of*

$$\Phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j \ (z \in \mathbb{C}):$$

1. $z_0 \in \mathbb{R}$ *a root with multiplicity 1:*
   $x_t = z_o^{-t}$;
2. $z_0 = re^{i\mu}$, $\overline{z}_0 = re^{-i\mu}$ *complex conjugate roots:*
   $x_t = \cos(\mu t) r^{-t}, \quad x_t = \sin(\mu t) r^{-t}$;
3. $z_0 \in \mathbb{R}$ *a root with multiplicity k:*
   $x_t = z_o^{-t} t^j \ (j = 0, \dots, k-1)$;
4. $z_0 = re^{i\mu}$, $\overline{z}_0 = re^{-i\mu}$ *each with multiplicity k:*
   $x_t = \cos(\mu t) r^{-t} t^j$,
   $x_t = \sin(\mu t) r^{-t} t^j \qquad (j = 0, \dots, k-1)$.

Sketch of proof.

The assertion that the solutions build a vector space holds because (1) is a homogeneous equation.

Regarding the dimensionality: consider $x_1, \ldots, x_p$ fixed: then, x_t is determined for all $t \in \mathbb{Z}$ since:

$$x_{p+1} = fct.(x_p, x_{p-1}, \ldots, x_1)$$

and we can then continue iteratively for all $x_t,\ t \geq p+1$; on the other hand,

$$x_0 = \frac{x_p - \sum_{j=1}^{p-1} \phi_j x_{p-j}}{\phi_p},$$

and we can continue iteratively for $x_t\ t \leq 0$.

The form of the basis vectors: if we plug them into (1) one can see that the equation holds.

Finally, it is "easy" to show that the basis vectors are linearly independent. □

Corollary 1. *Consider a causal, stationary AR(p) model. Then, every solution of the corresponding deterministic skeleton as in (1) converges exponentially fast to zero as* $t \to \infty$*.*

Proof.
A causal and stationary AR(p) must necessarily have its root of $\Phi(\cdot)$ outside the unit circle $\{z \in \mathbb{C};\ |z| \leq 1\}$. Therefore, the assertion holds since all roots have absolute value $|z_0| > 1$ and the solutions of (1) then decay exponentially fast, see Theorem 1. $\square$