state space formulation of ARMA (p,q)

consider

$$Y_t = \sum_{j=1}^{p} \phi_j Y_{t-j} + \sum_{h=1}^{q} \theta_h \varepsilon_{t-h} + \varepsilon_t$$

with $p > q$ ($p \le q$ is analogous)

Proposition

A causal (and stationary), invertible ARMA (p,q) with $p > q$ and with Gaussian innovations is a linear state space model with Gaussian errors:

$$Z_t = (Y_t, Y_{t+1|t}, \dots, Y_{t+p-1|t})$$

$$Y_{s|t} = \mathbb{E}[Y_s \mid Y_t, Y_{t-1}, \dots]$$

$$Y_t = \underbrace{[1, 0, \dots, 0]}_{=H} Z_t + \underset{W_t}{\overset{\parallel}{0}}$$

$$Z_t = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & & \\ 0 & 0 & 1 & 0 & \dots & \\ 0 & 0 & 0 & 1 & \dots & \\ \vdots & & & & \ddots & \\ 0 & & & & & 1 \\ \phi_p & \phi_{p-1} & \dots & & & \phi_1 \end{bmatrix}}_{=G} Z_{t-1} + \underbrace{\begin{bmatrix} \psi_0 = 1 \\ \psi_1 \\ \vdots \\ \\ \gamma \end{bmatrix}}_{=V_t} \varepsilon_t$$

"some" constant

<u>Proof</u> :

causality : $Y_t = \sum_{j=0}^{\infty} \psi_j \, \varepsilon_{t-j} \quad , \psi_0 = 1$ (1)

invertibility : $\varepsilon_t = \sum_{j=0}^{\infty} \gamma_j \, Y_{t-j} \quad , \gamma_0 = 1$ (2)

it holds :

$$Y_{t+i|t} = \sum_{j=1}^{\min(i-1,p)} \phi_j \, Y_{t+i-j|t} + \sum_{j=i}^{p} \phi_j \, Y_{t+i-j} + \sum_{j=i}^{q} \theta_j \, \varepsilon_{t+i-j}$$

(3)

then:

$$Y_{t+i|t+1} = E[Y_{t+i} | Y_{t+1}, \dots] \overset{(1)}{=} E\Big[\sum_{j=0}^{\infty} \psi_j \, \varepsilon_{t+i-j} \Big| Y_{t+1}, \dots\Big]$$

$$= \underbrace{E\Big[\sum_{j=0}^{i-2} \psi_j \, \varepsilon_{t+i-j} \Big| Y_{t+1}, \dots\Big]}_{\overset{(1)}{=} 0} + \underbrace{E\Big[\sum_{j=i-1}^{\infty} \psi_j \, \varepsilon_{t+i-j} \Big| Y_{t+1}, \dots\Big]}_{\overset{(2)}{=} \sum_{j=i-1}^{\infty} \psi_j \, \varepsilon_{t+i-j}}$$

$$= \sum_{j=i-1}^{\infty} \psi_j \, \varepsilon_{t+i-j}$$

(4)

$$Y_{t+i|t} = E\Big[\sum_{j=0}^{\infty} \psi_j\, \varepsilon_{t+i-j} \,\Big|\, Y_t, \dots\Big]$$

$$= \underbrace{E\Big[\sum_{j=0}^{i-1} \psi_j\, \varepsilon_{t+i-j} \,\Big|\, Y_t, \dots\Big]}_{=0} + \underbrace{E\Big[\sum_{j=i}^{\infty} \psi_j\, \varepsilon_{t+i-j} \,\Big|\, Y_t, \dots\Big]}_{\overset{(2)}{=} \sum_{j=i}^{\infty} \psi_j\, \varepsilon_{t+i-j}}$$

$$= \sum_{j=i}^{\infty} \psi_j\, \varepsilon_{t+i-j} \qquad (5)$$

Hence, by (4) and (5).

$$Y_{t+i|t+1} = Y_{t+i|t} + \psi_{i-1}\, \varepsilon_{t+1} \qquad (6)$$

Furthermore: $Y_{t+1} = Y_{t+1|t} + \varepsilon_{t+1}$ (7)

$$\xrightarrow{(6),(7)} \quad Z_{t+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & \cdots & 1 \\ \phi_p & \phi_{p-1} & \cdots & \cdots & \phi_1 \end{bmatrix} Z_t + \begin{bmatrix} \psi_0 = 1 \\ \psi_1 \\ \vdots \\ \psi_{p-2} \\ \gamma \end{bmatrix} \varepsilon_{t+1}$$

$$\xrightarrow{(3)}$$

$$Y_{t+p|t+1} = \sum_{j=1}^{p} \phi_j \underbrace{Y_{t+p-j|t+1}}_{\overset{(6)}{=} Y_{t+p-j|t} + \psi_{p-j-1}\, \varepsilon_{t+1}} + \phi_p Y_t$$