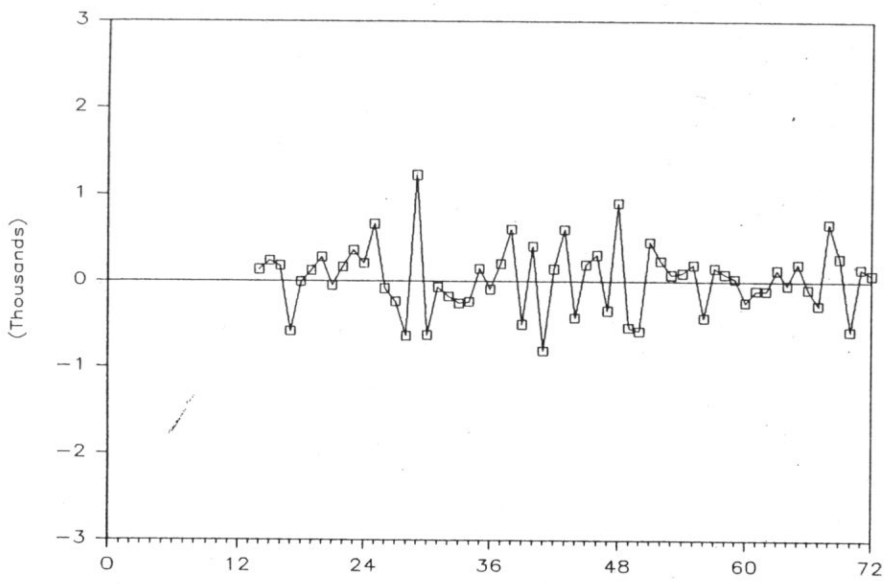

Figure 1.17. The differenced series $\{\nabla\nabla_{12}x_t, t = 14, \ldots, 72\}$ derived from the monthly accidental deaths $\{x_t, t = 1, \ldots, 72\}$.

Proposition 1.5.1 (Elementary Properties). *If $\gamma(\cdot)$ is the autocovariance function of a stationary process $\{X_t, t \in \mathbb{Z}\}$, then*

$$\gamma(0) \geq 0, \tag{1.5.1}$$

$$|\gamma(h)| \leq \gamma(0) \quad \text{for all } h \in \mathbb{Z}, \tag{1.5.2}$$

and

$$\gamma(h) = \gamma(-h) \quad \text{for all } h \in \mathbb{Z}. \tag{1.5.3}$$

PROOF. The first property is a statement of the obvious fact that $\text{Var}(X_t) \geq 0$, the second is an immediate consequence of the Cauchy–Schwarz inequality,

$$|\text{Cov}(X_{t+h}, X_t)| \leq (\text{Var}(X_{t+h}))^{1/2}(\text{Var}(X_t))^{1/2}$$

and the third is established by observing that

$$\gamma(-h) = \text{Cov}(X_{t-h}, X_t) = \text{Cov}(X_t, X_{t+h}) = \gamma(h). \qquad \square$$

Autocovariance functions also have the more subtle property of non-negative definiteness.

Definition 1.5.1 (Non-Negative Definiteness). A real-valued function on the integers, $\kappa : \mathbb{Z} \to \mathbb{R}$, is said to be non-negative definite if and only if

$$\sum_{i,j=1}^{n} a_i \kappa(t_i - t_j) a_j \geq 0 \tag{1.5.4}$$

for all positive integers n and for all vectors $\mathbf{a} = (a_1, \ldots, a_n)' \in \mathbb{R}^n$ and $\mathbf{t} = (t_1, \ldots, t_n)' \in \mathbb{Z}^n$.

Theorem 1.5.1 (Characterization of Autocovariance Functions). *A real-valued even function defined on the set $\mathbb{Z}$ of all integers is non-negative definite if and only if it is the autocovariance function of a stationary time series.*

PROOF. To show that the autocovariance function $\gamma(\cdot)$ of any stationary time series $\{X_t\}$ is non-negative definite, we simply observe that if $\mathbf{a} = (a_1, \ldots, a_n)' \in \mathbb{R}^n$, $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{Z}^n$, and $\mathbf{Z_t} = (X_{t_1} - EX_{t_1}, \ldots, X_{t_n} - EX_{t_n})'$, then

$$\begin{aligned} 0 &\leq \text{Var}(\mathbf{a}'\mathbf{Z_t}) \\ &= \mathbf{a}'E\mathbf{Z_t}\mathbf{Z_t}'\mathbf{a} \\ &= \mathbf{a}'\Gamma_n\mathbf{a} \\ &= \sum_{i,j=1}^{n} a_i \gamma(t_i - t_j) a_j, \end{aligned}$$

where $\Gamma_n = [\gamma(t_i - t_j)]_{i,j=1}^n$ is the covariance matrix of $(X_{t_1}, \ldots, X_{t_n})'$.

To establish the converse, let $\kappa : \mathbb{Z} \to \mathbb{R}$ be an even non-negative definite function. We need to show that there exists a stationary process with $\kappa(\cdot)$ as its autocovariance function, and for this we shall use Kolmogorov's theorem. For each positive integer n and each $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ such that $t_1 < t_2 < \cdots < t_n$, let $F_\mathbf{t}$ be the distribution function on $\mathbb{R}^n$ with characteristic function

$$\phi_\mathbf{t}(\mathbf{u}) = \exp(-\mathbf{u}'K\mathbf{u}/2),$$

where $\mathbf{u} = (u_1, \ldots, u_n)' \in \mathbb{R}^n$ and $K = [\kappa(t_i - t_j)]_{i,j=1}^n$. Since κ is non-negative definite, the matrix K is also non-negative definite and consequently $\phi_\mathbf{t}$ is the characteristic function of an n-variate normal distribution with mean zero and covariance matrix K (see Section 1.6). Clearly, in the notation of Theorem 1.2.1,

$$\phi_{\mathbf{t}(i)}(\mathbf{u}(i)) = \lim_{u_i \to 0} \phi_\mathbf{t}(\mathbf{u}) \quad \text{for each } \mathbf{t} \in \mathscr{T},$$

i.e. the distribution functions $F_\mathbf{t}$ are consistent, and so by Kolmogorov's theorem there exists a time series $\{X_t\}$ with distribution functions $F_\mathbf{t}$ and characteristic functions $\phi_\mathbf{t}$, $\mathbf{t} \in \mathscr{T}$. In particular the joint distribution of X_i and X_j is bivariate normal with mean $\mathbf{0}$ and covariance matrix

$$\begin{bmatrix} \kappa(0) & \kappa(i-j) \\ \kappa(i-j) & \kappa(0) \end{bmatrix},$$

which shows that $\text{Cov}(X_i, X_j) = \kappa(i - j)$ as required. $\square$

Remark 1. As shown in the proof of Theorem 1.5.1, for every autocovariance function $\gamma(\cdot)$, there exists a stationary Gaussian time series with $\gamma(\cdot)$ as its autocovariance function.

Remark 2. To verify that a given function is non-negative definite it is sometimes simpler to specify a stationary process with the given autocovariance function than to check Definition 1.4.1. For example the function $\kappa(h) = \cos(\theta h)$, $h \in \mathbb{Z}$, is the autocovariance function of the process in Example 1.3.1 and is therefore non-negative definite. Direct verification by means of Definition 1.4.1 however is more difficult. Another simple criterion for checking non- negative definiteness is Herglotz's theorem, which will be proved in Section 4.2.

Remark 3. An autocorrelation function $\rho(\cdot)$ has all the properties of an autocovariance function and satisfies the additional condition $\rho(0) = 1$.

EXAMPLE 1.5.1. Let us show that the real-valued function on Z,

$$\kappa(h) = \begin{cases} 1 & \text{if } h = 0, \\ \rho & \text{if } h = \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

is an autocovariance function if and only if $|\rho| \le \frac{1}{2}$.

If $|\rho| \le \frac{1}{2}$ then $\kappa(\cdot)$ is the autocovariance function of the process defined in Example 1.3.2 with $\sigma^2 = (1 + \theta^2)^{-1}$ and $\theta = (2\rho)^{-1}(1 \pm \sqrt{1 - 4\rho^2})$.

If $\rho > \frac{1}{2}$, $K = [\kappa(i - j)]_{i,j=1}^n$ and $\mathbf{a}$ is the n-component vector $\mathbf{a} = (1, -1, 1, -1, \ldots)'$, then

$$\mathbf{a}'K\mathbf{a} = n - 2(n-1)\rho < 0 \quad \text{for } n > 2\rho/(2\rho - 1),$$

which shows that $\kappa(\cdot)$ is not non-negative definite and therefore, by Theorem 1.5.1 is not an autocovariance function.

If $\rho < -\frac{1}{2}$, the same argument using the n-component vector $\mathbf{a} = (1, 1, 1, \ldots)$ again shows that $\kappa(\cdot)$ is not non-negative definite.

The Sample Autocovariance Function of an Observed Series

From the observations $\{x_1, x_2, \ldots, x_n\}$ of a stationary time series $\{X_t\}$ we frequently wish to estimate the autocovariance function $\gamma(\cdot)$ of the underlying process $\{X_t\}$ in order to gain information concerning its dependence structure. This is an important step towards constructing an appropriate mathematical model for the data. The estimate of $\gamma(\cdot)$ which we shall use is the sample autocovariance function.

Definition 1.5.2. The sample autocovariance function of $\{x_1, \ldots, x_n\}$ is defined by

$$\hat{\gamma}(h) := n^{-1} \sum_{j=1}^{n-h} (x_{j+h} - \bar{x})(x_j - \bar{x}), \qquad 0 \le h < n,$$

and $\hat{\gamma}(h) = \hat{\gamma}(-h)$, $-n < h \le 0$, where $\bar{x}$ is the sample mean $\bar{x} = n^{-1} \sum_{j=1}^n x_j$.

Remark 4. The divisor n is used rather than $(n - h)$ since this ensures that the matrix $\hat{\Gamma}_n := [\hat{\gamma}(i - j)]_{i,j=1}^n$ is non-negative definite (see Section 7.2).

Remark 5. The sample autocorrelation function is defined in terms of the sample autocovariance function as

$$\hat{\rho}(h) := \hat{\gamma}(h)/\hat{\gamma}(0), \qquad |h| < n.$$

The corresponding matrix $\hat{R}_n := [\hat{\rho}(i - j)]_{i,j=1}^n$ is then also non-negative definite.

Remark 6. The large-sample properties of the estimators $\hat{\gamma}(h)$ and $\hat{\rho}(h)$ are discussed in Chapter 7.

EXAMPLE 1.5.2. Figure 1.18(a) shows 300 simulated observations of the series $X_t = Z_t + \theta Z_{t-1}$ of Example 1.3.2 with $\theta = 0.95$ and $Z_t \sim N(0, 1)$. Figure 1.18(b) shows the corresponding sample autocorrelation function at lags $0, \ldots, 40$. Notice the similarity between $\hat{\rho}(\cdot)$ and the function $\rho(\cdot)$ computed as described in Example 1.3.2 ($\rho(h) = 1$ for $h = 0$, $.4993$ for $h = \pm 1$, 0 otherwise).

EXAMPLE 1.5.3. Figures 1.19(a) and 1.19(b) show simulated observations and the corresponding sample autocorrelation function for the process $X_t = Z_t + \theta Z_{t-1}$, this time with $\theta = -0.95$ and $Z_t \sim N(0, 1)$. The similarity between $\hat{\rho}(\cdot)$ and $\rho(\cdot)$ is again apparent.

Remark 7. Notice that the realization of Example 1.5.2 is less rapidly fluctuating than that of Example 1.5.3. This is to be expected from the two autocorrelation functions. Positive autocorrelation at lag 1 reflects a tendency for successive observations to lie on the same side of the mean, while negative autocorrelation at lag 1 reflects a tendency for successive observations to lie on opposite sides of the mean. Other properties of the sample-paths are also reflected in the autocorrelation (and sample autocorrelation) functions. For example the sample autocorrelation function of the Wölfer sunspot series (Figure 1.20) reflects the roughly periodic behaviour of the data (Figure 1.5).

Remark 8. The sample autocovariance and autocorrelation functions can be computed for *any* data set $\{x_1, \ldots, x_n\}$ and are not restricted to realizations of a stationary process. For data containing a trend, $|\hat{\rho}(h)|$ will exhibit slow decay as h increases, and for data with a substantial deterministic periodic component, $\hat{\rho}(h)$ will exhibit similar behaviour with the same periodicity. Thus $\hat{\rho}(\cdot)$ can be useful as an indicator of non-stationarity (see also Section 9.1).

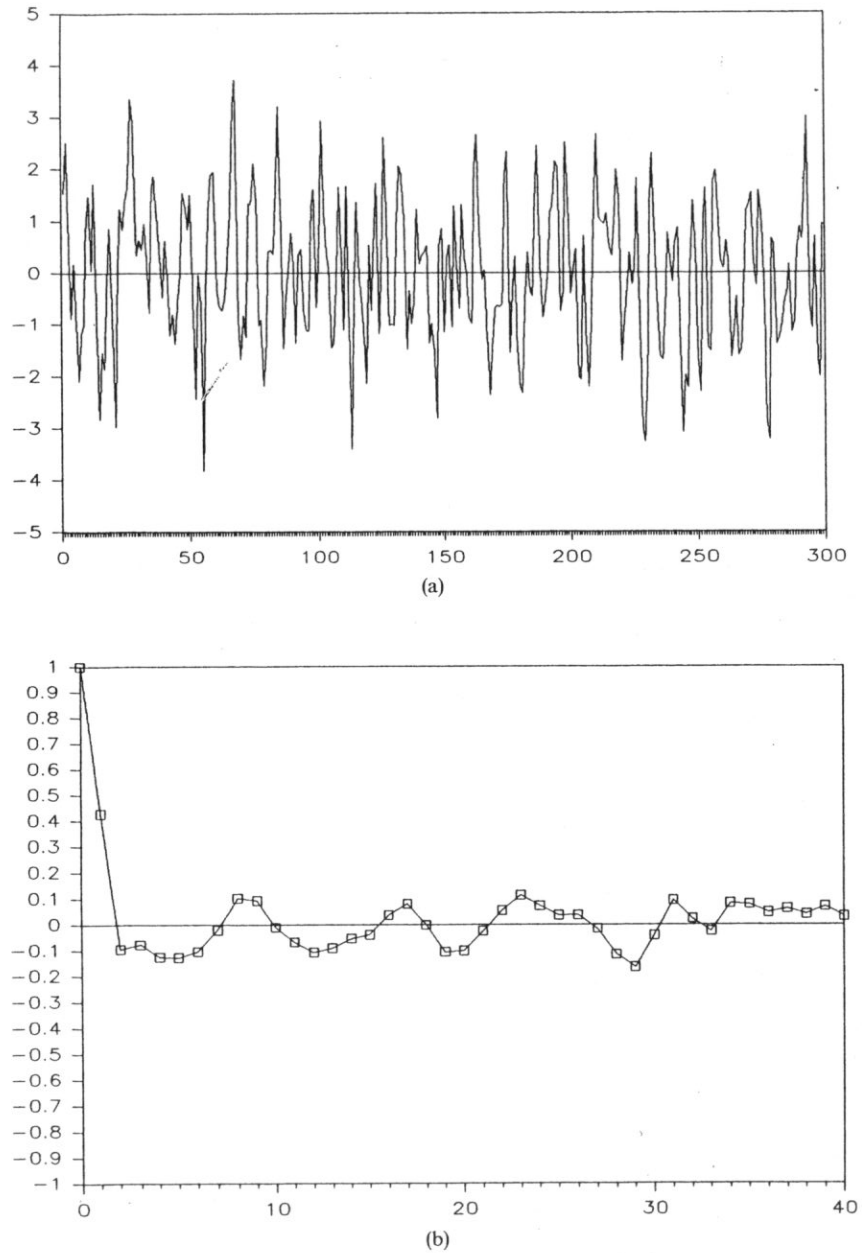

Figure 1.18. (a) 300 observations of the series $X_t = Z_t + .95Z_{t-1}$, Example 1.5.2. (b) The sample autocorrelation function $\hat{\rho}(h)$, $0 \le h \le 40$.

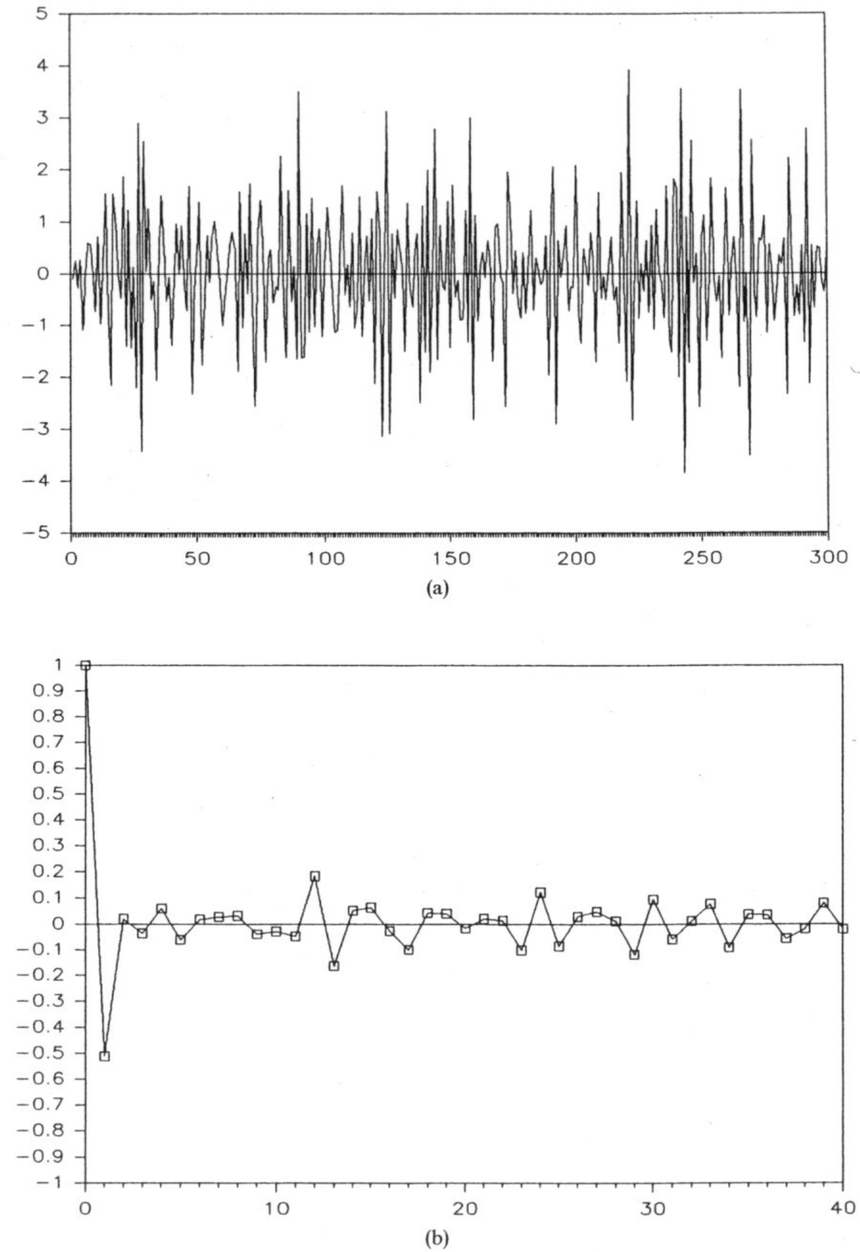

Figure 1.19. (a) 300 observations of the series $X_t = Z_t - .95Z_{t-1}$, Example 1.5.3. (b) The sample autocorrelation function $\hat{\rho}(h)$, $0 \le h \le 40$.