I.2. Central Limit Theorem for dependent observations

data: realizations of

$X_1, X_2, \dots, X_n$ stationary, ergodic process

consider estimator

$$\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$$

and we want a confidence interval
statistical test
standard error $\sqrt{\mathrm{Var}(\hat{\theta}_n)}$

based on $\hat{\theta}_n$

i.i.d. case

consider $\hat{\theta}_n = \overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$

$\hat{\theta}_n =$ MLE in parametric model

etc.

then: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2_\infty)$

$\sigma^2_\infty = \frac{1}{I(\theta)}$ (inverse of Fisher information) if $\hat{\theta}_n =$ MLE

$\sigma^2_\infty = \mathrm{Var}(X_1)$ if $\hat{\theta}_n = \overline{X}_n$

with dependence

consider $\hat{\theta}_n = \bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$

if $(X_t)_{t \in \mathbb{Z}}$ is stationary, ergodic proces and under additional regularity assumptions (e.g. mixing conditions):

$$\sqrt{n} \left(\bar{X}_n - E[X_1] \right) \xrightarrow{d} \mathcal{N}(0, \sigma_\infty^2)$$

$$\underbrace{\sigma_\infty^2 = \sum_{k=-\infty}^{\infty} Cov(X_0, X_k)}_{\text{an "infinite-dimensional object"}}$$

reasoning for σ_∞^2:

Lemma

Let $R(h) = Cov(X_0, X_k) \; \left(= Cov(X_t, X_{t+h}) \right)$

Assume $\sum_{h=-\infty}^{\infty} |R(h)| < \infty$. Then

$$n \, Var(\bar{X}_n) \longrightarrow \sum_{h=-\infty}^{\infty} R(h) = \sigma_\infty^2$$

Proof of Lemma:

$$n \operatorname{Var}(\bar{X}_n) = n \cdot \frac{1}{n^2} \sum_{t,s=1}^{n} \underbrace{\operatorname{Cov}(X_t, X_s)}_{= R(t-s)}$$

$$= \frac{1}{n} \Big(R(0)\cdot n + R(1)\cdot(n-1) + R(-1)\cdot(n-1) + \dots$$
$$+ R(n-1)\cdot 1 + R(-(n-1))\cdot 1 \Big)$$

$$\underset{R(h)=R(-h)}{=} \frac{1}{n} \sum_{h=-n+1}^{n-1} R(h)\cdot(n-|h|)$$

$$= \sum_{h=-n+1}^{n-1} R(h)\cdot\left(1 - \frac{|h|}{n}\right)$$

$$= \underbrace{\sum_{h=-n+1}^{n-1} R(h)}_{\to \sum_{h=-\infty}^{\infty} R(h)} - \underbrace{\frac{1}{n} \sum_{h=-n+1}^{n-1} R(h)\cdot|h|}_{\overset{|\cdot|}{\leq} \underbrace{\frac{1}{n}\sqrt{n}}_{\to 0} \sum_{h=-\lfloor\sqrt{n}\rfloor}^{\lfloor\sqrt{n}\rfloor} |R(h)| + \underbrace{2\sum_{h=\lfloor\sqrt{n}\rfloor+1}^{n-1} |R(h)|}_{\to 0}} \longrightarrow \sum_{h=-\infty}^{\infty} R(h)$$

$\blacksquare$

<u>discussion:</u>

statistical inference more difficult for dependent observations!

$$\sigma_\infty^2 = \sum_{h=-\infty}^{\infty} R(h)$$ is infinite-dimensional object

for i.i.d. case: $\sigma_\infty^2 = R(0) = Var(X_1)$

- estimation of σ_∞^2 is more difficult than estimation of $R(0) = Var(X_2)$
- $\hat{\sigma}_\infty^2$ via spectral analysis (see later) for estimating $\underbrace{\{R(k);\ h \in \mathbb{Z}\}}_{\text{covariance function}}$