

## Skeleton of AR( $p$ ) models

without innovations

$$x_t = \sum_{j=1}^p \phi_j x_{t-j}, \quad \phi_p \neq 0. \quad (1)$$

**Theorem 1.** *All solutions of (1) build a vector space of dimension  $p$ . The basis vectors correspond to the roots of*

$$\Phi(z) = 1 - \sum_{j=1}^p \phi_j z^j \quad (z \in \mathbb{C}) :$$

1.  $z_0 \in \mathbb{R}$  a root with multiplicity 1:  $x_t = z_0^{-t}$ ;
2.  $z_0 = re^{i\mu}$ ,  $\bar{z}_0 = re^{-i\mu}$  complex conjugate roots:  
 $x_t = \cos(\mu t)r^{-t}$ ,  $x_t = \sin(\mu t)r^{-t}$ ;
3.  $z_0 \in \mathbb{R}$  a root with multiplicity  $k$ :  $x_t = z_0^{-t} t^j$  ( $j = 0, \dots, k-1$ );
4.  $z_0 = re^{i\mu}$ ,  $\bar{z}_0 = re^{-i\mu}$  each with multiplicity  $k$ :  
 $x_t = \cos(\mu t)r^{-t} t^j$ ,  $x_t = \sin(\mu t)r^{-t} t^j$  ( $j = 0, \dots, k-1$ ).

*Sketch of proof.*

The assertion that the solutions build a vector space holds because (1) is a homogeneous equation.

Regarding the dimensionality: consider  $x_1, \dots, x_p$  fixed: then,  $x_t$  is determined for all  $t \in \mathbb{Z}$  since:

$$x_{p+1} = \text{fct.}(x_p, x_{p-1}, \dots, x_1)$$

and we can then continue iteratively for all  $x_t$ ,  $t \geq p+1$ ; on the other hand,

$$x_0 = \frac{x_p - \sum_{j=1}^{p-1} \phi_j x_{p-j}}{\phi_p},$$

and we can continue iteratively for  $x_t$   $t \leq 0$ .

The form of the basis vectors: if we plug them into (1) one can see that the equation holds.

Finally, it is “easy” to show that the basis vectors are linearly independent.  $\square$

**Corollary 1.** *Consider a causal, stationary AR( $p$ ) model. Then, every solution of the corresponding deterministic skeleton as in (1) converges exponentially fast to zero as  $t \rightarrow \infty$ .*

*Proof.*

A causal and stationary AR( $p$ ) must necessarily have its root of  $\Phi(\cdot)$  outside the unit circle  $\{z \in \mathbb{C}; |z| \leq 1\}$ . Therefore, the assertion holds since all roots have absolute value  $|z_0| > 1$  and the solutions of (1) then decay exponentially fast, see Theorem 1.  $\square$