Multi-step prediction

aim: prediction of X_{t+m-1} based on

$X_{t-1}, X_{t-2}, \dots \qquad (m \geq 1)$

assumption: $(X_t)_{t\in\mathbb{Z}}$ ARMA(p,q) with

$$\underbrace{\Phi(z) \neq 0, \quad \Theta(z) \neq 0}_{\text{different roots}} \quad (|z| \leq 1)$$

Then: $\quad X_t = \sum_{j=1}^{\infty} \gamma_j X_{t-j} + \varepsilon_t \quad (\varepsilon_t \text{ indep. of } \{X_s, s<t\})$

Consider $m \geq 2$: best prediction

$$\mathbb{E}[X_{t+m-1} | X_{t-1}, X_{t-2}, \dots]$$

$$= \sum_{j=1}^{m-1} \gamma_j \underbrace{\mathbb{E}[X_{t+m-1-j} | X_{t-1}, X_{t-2}, \dots]}_{= \hat{X}_{t+m-1-j|-\infty:(t-1)}} + \sum_{j=m}^{\infty} \gamma_j X_{t+m-1-j}$$

$$\Longrightarrow \underbrace{\hat{X}_{t+m-1|-\infty:(t-1)}}_{\text{linear in } X_{t-1}, X_{t-2}, \dots} = \sum_{j=1}^{m-1} \gamma_j \underbrace{\hat{X}_{t+m-1-j|-\infty:(t-1)}}_{\text{linear in } X_{t-1}, X_{t-2}, \dots} + \sum_{j=m}^{\infty} \gamma_j X_{t+m-1-j}$$

Corollary

$(X_t)_{t\in\mathbb{Z}}$ ARMA(p,q) with $\underbrace{\Phi(z)\neq 0,\ \Theta(z)=0}_{\text{different roots}}$ $(|z|\leq 1)$

Then: $\hat{X}_{t+m-1|-\infty:(t-1)} = \tilde{X}_{t+m-1|-\infty:(t-1)}$

i.e.: best prediction is linear

in practice:

$$\hat{X}_{t+m-1|-\infty:(t-1)} = \sum_{j=1}^{m-1} \phi_j \hat{X}_{t+m-1-j|-\infty:(t-1)}$$

$$+ \sum_{j=m}^{p} \phi_j X_{t+m-1-j} + \sum_{h=m}^{q} \theta_h \varepsilon_{t+m-1-h}$$

since $\mathbb{E}[\varepsilon_{t+m-1-h} | X_{t-1}, X_{t-2}, \dots] = 0$ $(0\leq h\leq m-1)$

implementation: recursively with starting values
(as one-step ahead prediction)

for $m > \max(p,q)$: $\underbrace{\hat{X}_{t+m-1|-\infty:(t-1)} = \sum_{j=1}^{p} \phi_j \hat{X}_{t+m-1-j|-\infty:(t-1)}}_{\text{det. AR}(p)\text{ skeleton}}$

$\rightsquigarrow \hat{X}_{t+m-1|-\infty:(t-1)} \xrightarrow{\text{exp.}} 0 = \mathbb{E}[X_t]$ as $m \to \infty$