

I.2. Central Limit Theorem for dependent observations

data: realizations of

X_1, X_2, \dots, X_n stationary, ergodic process

consider estimator

$$\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$$

and we want a confidence interval

statistical test

standard error $\sqrt{\text{Var}(\hat{\theta}_n)}$

based on $\hat{\theta}_n$

i.i.d. case

consider $\hat{\theta}_n = \bar{X}_n = \frac{1}{n} \sum_{t=1}^n X_t$

$\hat{\theta}_n = \text{MLE}$ in parametric model
etc.

then: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma_\infty^2)$

$\sigma_\infty^2 = \frac{1}{I(\theta)}$ (inverse of Fisher information)
if $\hat{\theta}_n = \text{MLE}$

$\sigma_\infty^2 = \text{Var}(X_1)$ if $\hat{\theta}_n = \bar{X}_n$

With dependence

consider $\hat{\Theta}_n = \bar{X}_n = \frac{1}{n} \sum_{t=1}^n X_t$

if $(X_t)_{t \in \mathbb{Z}}$ is stationary, ergodic process and under additional regularity assumptions (e.g. mixing conditions):

$$\sqrt{n} (\bar{X}_n - E[X_1]) \xrightarrow{d} \mathcal{N}(0, \sigma_\infty^2)$$

$$\sigma_\infty^2 = \sum_{k=-\infty}^{\infty} \text{Cov}(X_0, X_k)$$

an "infinite-dimensional object"

reasoning for σ_∞^2 :

Lemma

Let $R(k) = \text{Cov}(X_0, X_k) (= \text{Cov}(X_t, X_{t+k}))$

Assume $\sum_{k=-\infty}^{\infty} |R(k)| < \infty$. Then

$$n \text{Var}(\bar{X}_n) \longrightarrow \sum_{k=-\infty}^{\infty} R(k) = \sigma_\infty^2$$

Proof of Lemma:

$$\begin{aligned}n \operatorname{Var}(\bar{X}_n) &= n \cdot \frac{1}{n^2} \sum_{t,s=1}^n \underbrace{\operatorname{Cov}(X_t, X_s)}_{= R(t-s)} \\&= \frac{1}{n} \left(R(0) \cdot n + R(1) \cdot (n-1) + R(-1) \cdot (n-1) + \dots \right. \\&\quad \left. + R(n-1) \cdot 1 + R(-(n-1)) \cdot 1 \right) \\&= \frac{1}{n} \sum_{h=-n+1}^{n-1} R(h) \cdot (n - |h|) \\R(h) &= R(-h) \\&= \sum_{h=-n+1}^{n-1} R(h) \cdot \left(1 - \frac{|h|}{n}\right) \\&= \underbrace{\sum_{h=-n+1}^{n-1} R(h)}_{\rightarrow \sum_{h=-\infty}^{\infty} R(h)} - \underbrace{\frac{1}{n} \sum_{h=-n+1}^{n-1} R(h) \cdot |h|}_{\begin{aligned} &\leq \frac{1}{n} \sqrt{n} \sum_{h=-\lfloor \sqrt{n} \rfloor}^{\lfloor \sqrt{n} \rfloor} |R(h)| \rightarrow 0 \\ &+ 2 \sum_{h=\lfloor \sqrt{n} \rfloor+1}^{n-1} |R(h)| \rightarrow 0 \end{aligned}} \rightarrow \sum_{h=-\infty}^{\infty} R(h)\end{aligned}$$

□

discussion:

statistical inference more difficult for dependent observations!

$$\sigma_{\infty}^2 = \sum_{h=-\infty}^{\infty} R(h) \quad \text{is infinite-dimensional object}$$

for i.i.d. case: $\sigma_{\infty}^2 = R(0) = \text{Var}(X_1)$

- estimation of σ_{∞}^2 is more difficult than estimation of $R(0) = \text{Var}(X_1)$
 - $\hat{\sigma}_{\infty}^2$ via spectral analysis (see later)
- for estimating $\underbrace{\{R(h); h \in \mathbb{Z}\}}_{\text{covariance function}}$