Inverse problems

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The deconvolution problem

Suppose that we have nonnegative observations $Z_1; : : : ; Z_n$ from a distribution with density

$$h_0(z) = \int g(z - x)\, dF_0(x); \; z \geq 0;$$

where g is a known decreasing continuous density on $[0; \infty)$ and F_0 is the distribution function we want to estimate.

F_0 has support, contained in $[0; \infty)$ (i.e., corresponds to nonnegative random variables).

The maximum likelihood estimator (MLE) $\hat{F}_n$ of F_0 is obtained by maximizing the log likelihood

$$\sum_{i=1}^{n} \log \int g(Z_i - x)\, dF(x);$$

over all distribution functions F.

Conjecture in part 2 of Groeneboom and Wellner (1992): at an interior point t of the support of F_0:

$$n^{1=3} \left\{ \hat{F}_n(t) - F_0(t) \right\} \longrightarrow cZ;$$

where Z is the location of the minimum of 2-sided Brownian motion plus a parabola.

An important step in proving the conjectured behavior in Groeneboom and Wellner (1992) is to write the functional

$$\int_{x\in[0;t)} g(t-x)\,d\hat{F}_n(x)$$

in the form

$$g(0)\hat{F}_n(t) - \int_{x\in[0;t)} \{g(t-x)-g(0)\}\,d\hat{F}_n(x); \tag{1}$$

and to show that a centered version of $\int_{x\in[0;t)}\{g(t-x)-g(0)\}\,d\hat{F}_n(x)$ is of lower order than $g(0)\hat{F}_n(t)$: $\int_{x\in[0;t)}\{g(t-x)-g(0)\}\,d\hat{F}_n(x)$ is a so-called smooth functional. Note that

$$g(0)\hat{F}_n(t) = \int_{x\in[0;t)} g(0)\,d\hat{F}_n(x);$$

and that the only crucial difference of the latter integral with the integral in (1) is that the integrand of the integral in (1) is continuous at $x=t$. We want in fact to prove that

$$\int_{x\in[0;t)}\{g(t-x)-g(0)\}\,d\hat{F}_n(x) - \int_{x\in[0;t)}\{g(t-x)-g(0)\}\,dF_0(x) = O_p\left(n^{-1=2}\right);$$

whereas $\hat{F}_n(t)$ itself will have the so-called "cube root" behavior.

Integral equations

Canonical approach: consider the functional

$$K_t(F) = \int_{x\in[0;t)} \{g(t-x) - g(0)\}\, dF(x);$$

and let $a_{t;F}$ solve the (adjoint, see below) equation

$$\begin{aligned} \left[L_F^*(a)\right](x) &= E\left\{a_{t;F}(Z) \mid X = x\right\} \\ &= \int_{z\geq x} a_{t;F}(z) g(z-x)\, dz = \{g(t-x) - g(0)\} 1_{[0;t)}(x) - K_t(F); \end{aligned} \tag{2}$$

where $a_{t;F}$ has to be in the range of the score operator:

$$a_{t;F}(z) = \left[L_F(b)\right](z) = E_F\left\{b_{t;F}(X) \mid X + Y = z\right\} = \frac{\int_{[0;z]} b_{t;F}(x) g(z-x)\, dF(x)}{h_F(z)}: \tag{3}$$

If we could solve these equations for $\hat{F}_n$, we would have a representation of the following form:

$$K_t(\hat{F}_n) - K_t(F_0) = \int a_{t;\hat{F}_n}(z)\, d\left(H_n - H_0\right)(z):$$

"Argument":

$$\int a_{t;\hat F_n}(z)\,dH_n(z) = \int_{x\in[0,\infty)} \frac{\int_{x\in[0,z]} g(z-x)\,b_{t;\hat F_n}(x)\,d\hat F_n(x)}{h_{\hat F_n}(z)}\,dH_n(z)$$
$$= \int_{x\in[0,\infty)}\int_{z\ge x}\frac{g(z-x)}{\hat h_n(z)}\,dH_n(z)\,b_{t;\hat F_n}(x)\,d\hat F_n(x) = \int_{x\in[0,\infty)} b_{t;\hat F_n}(x)\,d\hat F_n(x) = 0;$$

and

$$\int a_{t;\hat F_n}(z)\,dH_0(z) = \int a_{t;\hat F_n}(z)\int_0^z g(z-x)\,dF_0(x)\,dz$$
$$= \int_{x\in[0,\infty)}\int_{z\ge x} a_{t;\hat F_n}(z)g(z-x)\,dz\,dF_0(x)$$
$$= \int_{x\in[0,\infty)}\left\{\{g(z-x)-g(0)\}1_{[0,t)}(x) - K_t(\hat F_n)\right\}\,dF_0(x)$$
$$= \int_{x\in[0,\infty)}\{g(z-x)-g(0)\}1_{[0,t)}(x)\,dF_0(x) - K_t(\hat F_n)$$
$$= K_t(F_0) - K_t(\hat F_n).$$

Unfortunately, there is generally no $b_{t;\hat F_n}$ such that

$$a_{t;\hat F_n}(z) = E_{\hat F_n}\left\{b_{t;\hat F_n}(X)\mid X+Y=z\right\} = \frac{\int_{[0,z]} b_{t;\hat F_n}(x)g(z-x)\,d\hat F_n(x)}{h_{\hat F_n}(z)}.$$

Solution (first step): We introduce a right-continuous function $B_{t;\hat{F}_n}$ such that

$$\frac{\int_{[0;z]} g(z-x)\,dB_{t;\hat{F}_n}(x)}{h_{\hat{F}_n}(z)} = a_{t;\hat{F}_n}(z); \qquad \lim_{x\to\infty} B_{t;\hat{F}_n}(x) = 0;$$

where $B_{t;\hat{F}_n}$ is no longer absolutely continuous w.r.t. $\hat{F}_n$ and try again:

$$\int a_{t;\hat{F}_n}(z)\,dH_n(z) = \int_{x\in[0;\infty)} \frac{\int_{x\in[0;z]} g(z-x)\,b_{t;\hat{F}_n}(x)\,d\hat{F}_n(x)}{h_{\hat{F}_n}(z)}\,dH_n(z)$$
$$= \int_{x\in[0;\infty)} \int_{z\geq x} \frac{g(z-x)}{\hat{h}_n(z)}\,dH_n(z)\,dB_{t;\hat{F}_n}(x) \stackrel{?}{=} 0$$

Difficulty: characterization of MLE $\hat{F}_n$ tells us that

$$\int_{z\geq x} \frac{g(z-x)}{\hat{h}_n(z)}\,dH_n(z) \quad \begin{cases} \geq 1; \\ = 1; \text{ if } x \text{ is a point of mass of } \hat{F}_n. \end{cases}$$

Solution (second step): Introduce a function $\bar{B}_{t;\hat{F}_n}$ that is constant on the same intervals as $\hat{F}_n$ and equal to $B_{t;\hat{F}_n}$ at points of mass of $\hat{F}_n$. Then:

$$\int_{x\in[0;\infty)} \int_{z\geq x} \frac{g(z-x)}{\hat{h}_n(z)}\,dH_n(z)\,d\bar{B}_{t;\hat{F}_n}(x) = \lim_{x\to\infty} \bar{B}_{t;\hat{F}_n}(x) = \lim_{x\to\infty} B_{t;\hat{F}_n}(x) = 0;$$

and, hopefully, the following difference will be "small":

$$\int_{x\in[0;\infty)} \int_{z\geq x} \frac{g(z-x)}{\hat{h}_n(z)}\,dH_n(z)\,d\bar{B}_{t;\hat{F}_n}(x) - \int_{x\in[0;\infty)} \int_{z\geq x} \frac{g(z-x)}{\hat{h}_n(z)}\,dH_n(z)\,dB_{t;\hat{F}_n}(x).$$

Now:

$$K_t(\hat{F}_n) - K_t(F_0) = \int_{[0,t)} \{g(t-x) - g(0)\}\, d\hat{F}_n(x) - \int_{[0,t)} \{g(t-x) - g(0)\}\, dF_0(x)$$

$$= \int a_{t,\hat{F}_n}(z)\, d\left(H_n - H_0\right)(z) + \int_{x\in[0,\infty)} \int_{z\geq x} \frac{g(z-x)}{\hat{h}_n(z)}\, dH_n(z)\, d\left(\bar{B}_{t,\hat{F}_n} - B_{t,\hat{F}_n}\right)(x).$$

Solution (third step): Prove that

$$\int a_{t,\hat{F}_n}(z)\, d\left(H_n - H_0\right)(z) = \int a_{t,F_0}(z)\, d\left(H_n - H_0\right)(z) + o_p(n^{-1/2}).$$

That's the general plan!

Example

We consider $g(x) = 4(1-x)^3 1_{[0;1]}(x)$ and F_0 the Uniform$(0;1)$ distribution. Then we get the following equation in $\;_{t;F_0}(z) \stackrel{\text{def}}{=} \int_{x\in[0;z)} g(z-x)\, dB_{t;F_0}(x)$:

$$\int_{z=x}^{x+1} a_{t;F_0}(z)\, g(z-x)\, dz = \int_{z=x}^{x+1} \frac{\;_{t;F_0}(z)}{h_0(z)}\, g(z-x)\, dz = \{g(t-x)-g(0)\}1_{[0;t)}(x) - K_t(F_0): \tag{4}$$

Writing $\;=\;_{t;F_0}$ and $B = B_{t;F_0}$ and $a(x) = \;(x)=h_0(x)$ we get by differentiating:

$$-4a(x) + 12\int_{z=x}^{x+1} a(z)(1+x-z)^2\, dz = 12(1-t+x)^2 \cdot 1_{[0;t)}(x);\; x \neq t; \tag{5}$$

which leads to the following integral equation, using $B(1) = 0$,

$$\frac{B(x) - 3\int_0^x (1+u-x)^2 B(u)\, du}{h_0(x)} - 3\int_{z=x}^{1} \frac{B(z)(1+x-z)^2}{h_0(z)}\, dz$$

$$+9\int_{u=0}^{x} B(u) \int_{z=x}^{1+u} \frac{(1+u-z)^2(1+x-z)^2 dz}{h_0(z)}\, du$$

$$+9\int_{u=x}^{1} B(u) \int_{z=u}^{1+x} \frac{(1+u-z)^2(1+x-z)^2\, dz}{h_0(z)}\, du$$

$$= -\tfrac{3}{4}(1-t+x)^2 \cdot 1_{[0;t)}(x);\; x \neq t:$$

We can also write this integral equation in the following form:

$$\begin{aligned} B(x) - 3\int_0^x B(u)(1+u-x)^2\,du - 3h_0(x)\int_{u=x}^1 \frac{B(u)(1+x-u)^2}{h_0(u)}\,du \\ + 9h_0(x)\int_{u=0}^x B(u)\int_{z=x}^{1+u} \frac{(1+u-z)^2(1+x-z)^2\,dz}{h_0(z)}\,du \\ + 9h_0(x)\int_{u=x}^1 B(u)\int_{z=u}^{1+x} \frac{(1+u-z)^2(1+x-z)^2\,dz}{h_0(z)}\,du \\ = -\tfrac{3}{4}(1-t+x)^2 h_0(x)\cdot 1_{[0;t)}(x);\ x \neq t: \end{aligned} \tag{6}$$

Introducing the notation

$$C_{t;F_0}(x) = C(x) = \frac{B(x)}{h_0(x)};$$

this can also be written as an integral equation in $C(x)$:

$$\begin{aligned} C(x) - \frac{3}{h_0(x)}\int_0^x C(u)(1+u-x)^2\,dH_0(u) - 3\int_{u=x}^1 C(u)(1+x-u)^2\,du \\ + 9\int_{u=0}^x C(u)\int_{z=x}^{1+u} \frac{(1+u-z)^2(1+x-z)^2\,dz}{h_0(z)}\,dH_0(u) \\ + 9\int_{u=x}^1 C(u)\int_{z=u}^{1+x} \frac{(1+u-z)^2(1+x-z)^2\,dz}{h_0(z)}\,dH_0(u) \\ = -\tfrac{3}{4}(1-t+x)^2\cdot 1_{[0;t)}(x);\ x \neq t: \end{aligned} \tag{7}$$

Lemma 1. *Let $B_{t;F_0}$ and $C_{t;F_0}$ be the solutions of the integral equation (6) and (7), respectively.*

(i) $B_{t;F_0}$ *is non-positive, bounded and continuous on* $[0,1]$ *and* $B_{t;F_0}(0) = B_{t;F_0}(1) = 0$. *Moreover,* $B_{t;F_0}$ *has a bounded derivative at each point* $x \in (0,1) \setminus \{t\}$, *a jump of size* $\frac{3}{4}h_0(t)$ *at* t, *a finite right derivative at* $x = 0$ *and a left derivative, equal to zero, at* $x = 1$.

(ii) $C_{t;F_0}$ *is non-positive and bounded on* $(0,1)$ *with a bounded right limit at* 0 *and a left limit, equal to zero, at* 1. *Moreover,* $C_{t;F_0}$ *has a bounded derivative at each point* $x \in (0,1) \setminus \{t\}$, *a jump of size* $3/4$ *at* t, *a finite right derivative at* $x = 0$ *and a left derivative, equal to zero, at* $x = 1$.

(iii) $a_{t;F_0}$ *is bounded on* $(0,2)$ *with a bounded right limit at* 0 *and a left limit, equal to zero, at* 2. *Moreover,* $a_{t;F_0}$ *has a bounded derivative at each point* $z \in (0,2) \setminus \{t\}$, *a jump of size* $3/2$ *at* t, *and finite right and left derivatives at* $z = 0$ *and* $z = 2$, *respectively.*

The current status model

"Hidden space" variables are (T_i, X_i), $T_i, X_i \in \mathbb{R}$, observations are: (T_i, Δ_i).

X_i is independent of T_i, $\Delta_i = 1_{\{X_i \le T_i\}}$. The X_i are (unobservable) "failure times".

(Relevant part of) Log likelihood for the distribution function F of X_i:

$$\sum_{i=1}^{n} \left\{\Delta_i \log F(T_i) + (1-\Delta_i)\log\left(1-F(T_i)\right)\right\}. \tag{8}$$

Define the empirical processes:

$$V_{n1}(t) = n^{-1}\sum_{T_i \le t} \Delta_i, \qquad V_{n2}(t) = n^{-1}\sum_{T_i \le t}(1-\Delta_i), \qquad t \in \mathbb{R},$$

Then the log likelihood (8) for F, divided by n, can be written:

$$\int \log F(u)\, dV_{n1}(u) + \int \log\{1-F(u)\}\, dV_{n2}(u). \tag{9}$$

How can we determine the local behavior?

Problem: Unlike in $\sqrt{n}$-asymptotics, we do not have global convergence of the (rescaled) log likelihood process.

The situation is therefore fundamentally different from (1-dimensional) right-censoring, where for example the Kaplan-Meier estimator converges at $\sqrt{n}$-rate and maximizes a process which converges globally after rescaling.

But in the current situation we are lucky: **convex minorant interpretation of the MLE**.

Proposition 1. *Let H_n be the* **greatest convex minorant** *of the (so-called)* **cusum diagram** (or **cumulative sum diagram**), *consisting of the set of points*

$$\mathcal{P}_n = \left\{\left(\mathbb{G}_n(t); V_{n1}(t)\right); \; t \in \mathbb{R}\right\}; \; V_{n1}(t) = n^{-1}\sum_{i=1}^{n} \Delta_i 1_{(-\infty;t]}(T_i) \tag{10}$$

where $\mathbb{G}_n(t) = n^{-1}\sum_{i=1}^{n} 1_{(-\infty;t]}(T_i)$ *is the empirical distribution function of the observation times* $T_1;:::;T_n$.

Then $\hat{F}_n$ *is an MLE if and only if, at each observation point* $t = T_i$, $\hat{F}_n(t)$ **is the left derivative of** H_n *at* $\mathbb{G}_n(t)$. $\hat{F}_n$ *is uniquely determined at each observation point* T_i.

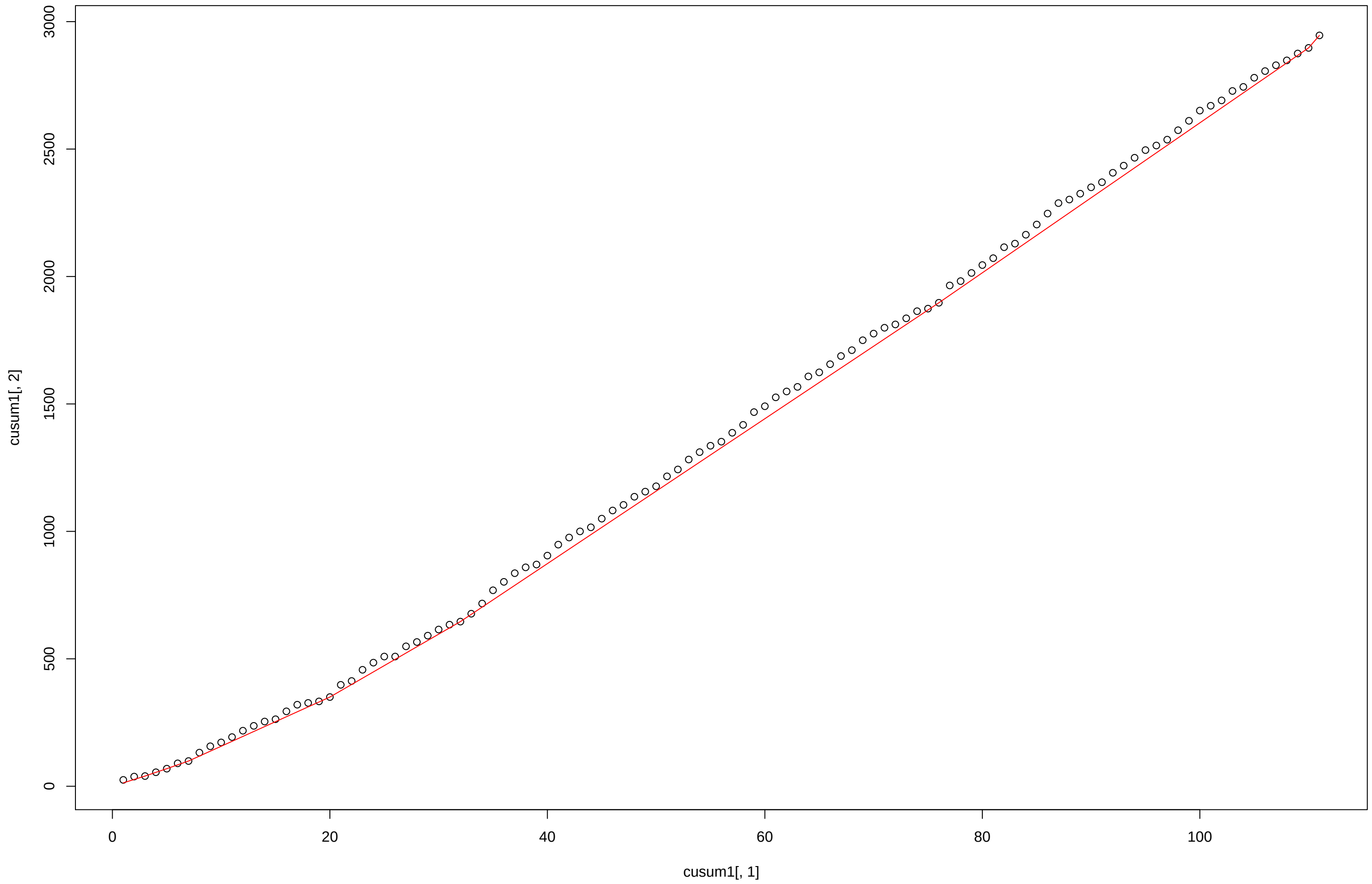

Figure 1: Cusum diagram. Simulation for $n = 20$. Observation df G of the T_i and df F of the X_i are both uniform.

Local asymptotic behavior

Define

$$W_n(t) = n^{-1}\sum_{i=1}^{n}\{\Delta_i - F_0(T_i)\}\left\{1_{\{T_i\le t\}} - 1_{\{T_i\le t_0\}}\right\},\ t\in\mathbb{R}. \tag{11}$$

Let t_0 be such that $0 < F_0(t_0), G(t_0) < 1$, and let F_0 and G be continuously differentiable at t_0, with strictly positive derivatives $f_0(t_0)$ and $g(t_0)$, respectively.

Then we have a "Kim and Pollard (1990)-type lemma":

$$W_n(t) = O_p\left(n^{-2/3}\right) + o_p\left((t-t_0)^2\right),\ \text{uniformly for } |t-t_0| \le \ . \tag{12}$$

After rescaling, the MLE $\hat F_n$ is the **slope of the convex minorant** of the process

$$U_n(t) \stackrel{\text{def}}{=} n^{2/3}W_n\left(t_0+n^{-1/3}t\right) + n^{-1/3}\sum_{i=1}^{n}\{F_0(T_i)-F_0(t_0)\}\left\{1_{\{T_i\le t\}} - 1_{\{T_i\le t_0\}}\right\},\ t\in\mathbb{R},$$

which converges to two-sided (scaled) Brownian motion with a parabolic drift. We can localize, due to the fact that, for large $|t|$, the drift in the process U_n is dominated by the parabolic drift of

$$n^{-1/3}\sum_{i=1}^{n}\{F_0(T_i)-F_0(t_0)\}\left\{1_{\{T_i\le t\}} - 1_{\{T_i\le t_0\}}\right\} \sim \tfrac{1}{2}f_0(t_0)t^2,\ n\to\infty.$$

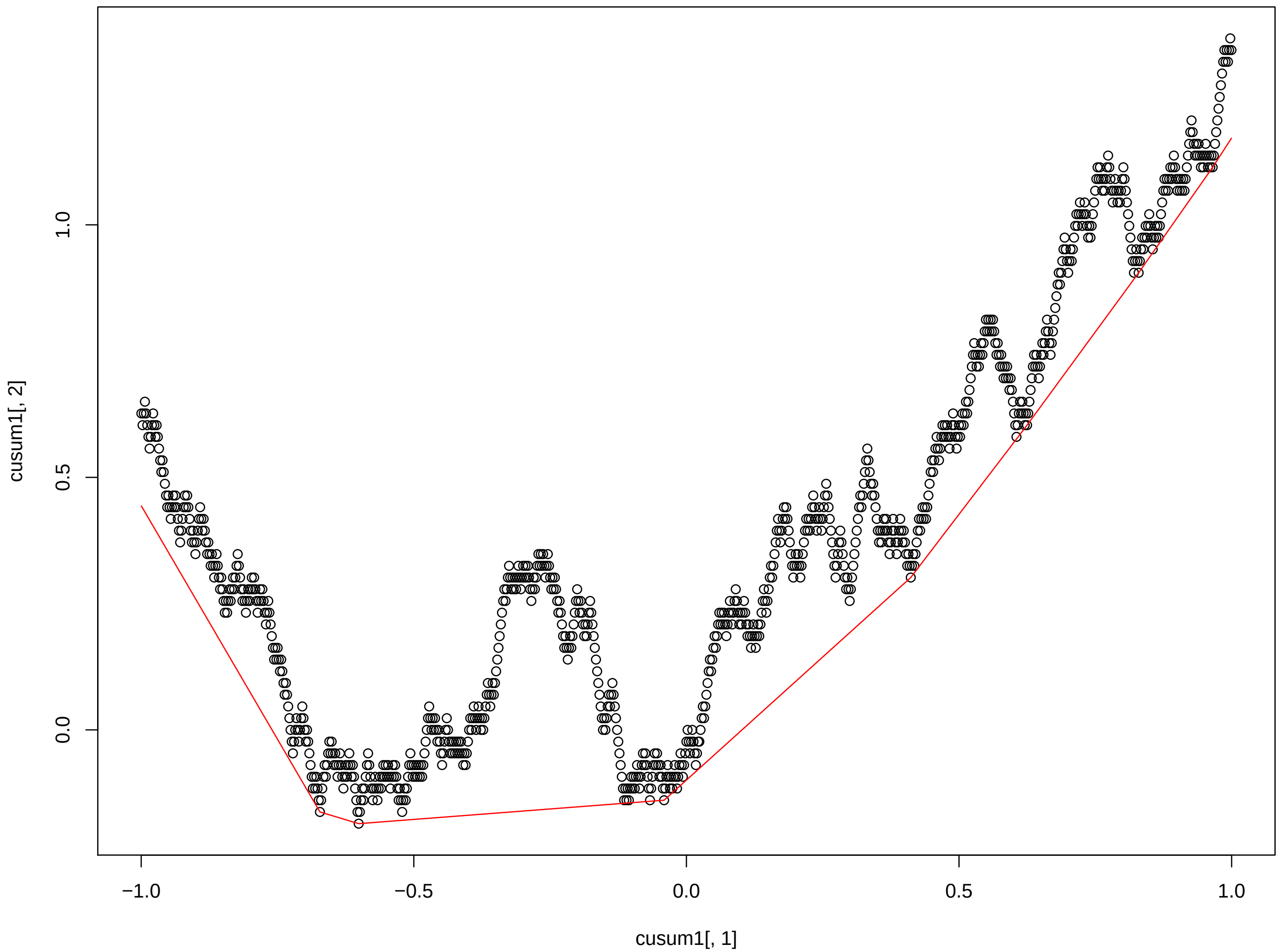

Figure 2: **Locally rescaled cusum process** $\left(n^{1/3}\left(\mathbb{G}_n(t_0+n^{1/3}t)-\mathbb{G}_n(t_0)\right);U_n(t)\right)$, **with convex minorant, for** $n^{1/3}|\mathbb{G}_n(t_0+n^{1/3}t)-\mathbb{G}_n(t_0)|\le 1$ **and** $t_0=0{:}5$. Simulation with $n=10{;}000$. Observation df G of the T_i and df F of the X_i are both **uniform**.

Local limit distribution of MLE $\hat{F}_n$

Using the "Kim and Pollard (1990)-type lemma":

$$W_n(t) = O_p\left(n^{-2=3}\right) + o_p\left((t-t_0)^2\right); \text{ uniformly for } |t-t_0| \le \ ;$$

we can "localize" the convex minorant and hence its derivative process, yielding the MLE $\hat{F}_n$.

Sketch of derivation of local limit distribution:

1. The localized cusum diagram

$$\left(n^{1=3}\left(\mathbb{G}_n(t_0 + n^{1=3}t) - t_0\right); U_n(t)\right)$$

converges in distribution to the Brownian motion cusum diagram:

$$(g(t_0)t; U(t)); \text{ where } U(t) = \sqrt{g(t_0)F_0(t_0)\{1-F_0(t_0)\}}\, W(t) + \tfrac{1}{2} f_0(t_0)g(t_0)t^2; \ t \in \mathbb{R};$$

and where W is two-sided Brownian motion.

2. Continuous mapping theorem: Convex minorant of localized cusum diagram converges in distribution to convex minorant of Brownian cusum diagram.

3. Left-derivative of convex minorant of localized cusum diagram converges in distribution (in Skohorod topology) to left-derivative of convex minorant of Brownian cusum diagram and $n^{1=3}\{\hat{F}_n(t_0) - F_0(t_0)\}$ **is left-derivative of convex minorant of localized cusum diagram at zero**.

Competing risk model

Generalization of the current status model to the situation where there are more failure causes.

Hidden space variables are (T_i, X_i, Y_i), Y_i is the failure cause.

Observations are: $(T_i, \Delta_{i1}, \dots, \Delta_{iK})$, $\Delta_{ik} = 1_{\{X_i \le T_i, Y_i = k\}}$. Define:

$$V_{nk}(t) = n^{-1} \sum_{i=1}^{n} \Delta_{ik} 1_{(-\infty,t]}(T_i), \; V_{n,K+1}(t) = n^{-1} \sum_{i=1}^{n} (1 - \Delta_{i+}) \, 1_{(-\infty,t]}(T_i), \; \Delta_{i+} = \sum_{k=1}^{K} \Delta_{ik},$$

We want to estimate the subdistribution functions F_{0k}:

$$F_{0k}(t) = P\{X \le t, Y = k\}, \; k = 1, \dots, K.$$

The (relevant part of the) log likelihood for $F = (F_1, \dots, F_K)$, divided by n, is:

$$\sum_{k=1}^{K} \int \log F_k(u) \, dV_{nk}(u) + \int \log\{1 - F_+(u)\} \, dV_{n,K+1}(u), \qquad F_+ = \sum_{k=1}^{K} F_k.$$

The MLE (maximum likelihood estimator) $\hat{F}_n = (\hat{F}_{n1}, \dots, \hat{F}_{nK})$ can only be computed iteratively.

No direct convex minorant interpretation, as with the MLE for current status data.

Self-induced characterization

The MLE $\hat{F}_{nk}$ can be characterized as the left derivative of the greatest convex minorant of the self-induced cusum diagram

$$\mathcal{P}_{nk} = \left\{ \left(G_{\hat{F}_{n+}}(t), V_{nk}(t) \right), \ t \in \mathbb{R} \right\}, \ V_{nk}(t) = n^{-1} \sum_{i=1}^{n} \Delta_{ik} 1_{(-\infty,t]}(T_i), \tag{13}$$

for $k = 1, \ldots, K$, where

$$G_{\hat{F}_{n+}}(t) = n^{-1} \sum_{i=1}^{n} \frac{1 - \Delta_{i+}}{1 - \hat{F}_{n+}(T_i)} 1_{(-\infty,t]}(T_i), \ t < T_{(n)}.$$

Compare with ordinary current status, where $\hat{F}_n$ is the left derivative of the greatest convex minorant of the (not self-induced) cusum diagram

$$\mathcal{P}_{n} = \left\{ \left(\mathbb{G}_n(t), V_{n1}(t) \right), \ t \in \mathbb{R} \right\}, \ V_{n1}(t) = n^{-1} \sum_{i=1}^{n} \Delta_{i} 1_{(-\infty,t]}(T_i). \tag{14}$$

Note:

$$G_{\hat{F}_{n+}}(t) = \mathbb{G}_n(t) + n^{-1} \sum_{i=1}^{n} \frac{\hat{F}_{n+}(T_i) - \Delta_{i+}}{1 - \hat{F}_{n+}(T_i)} 1_{(-\infty,t]}(T_i), \ t < T_{(n)}.$$

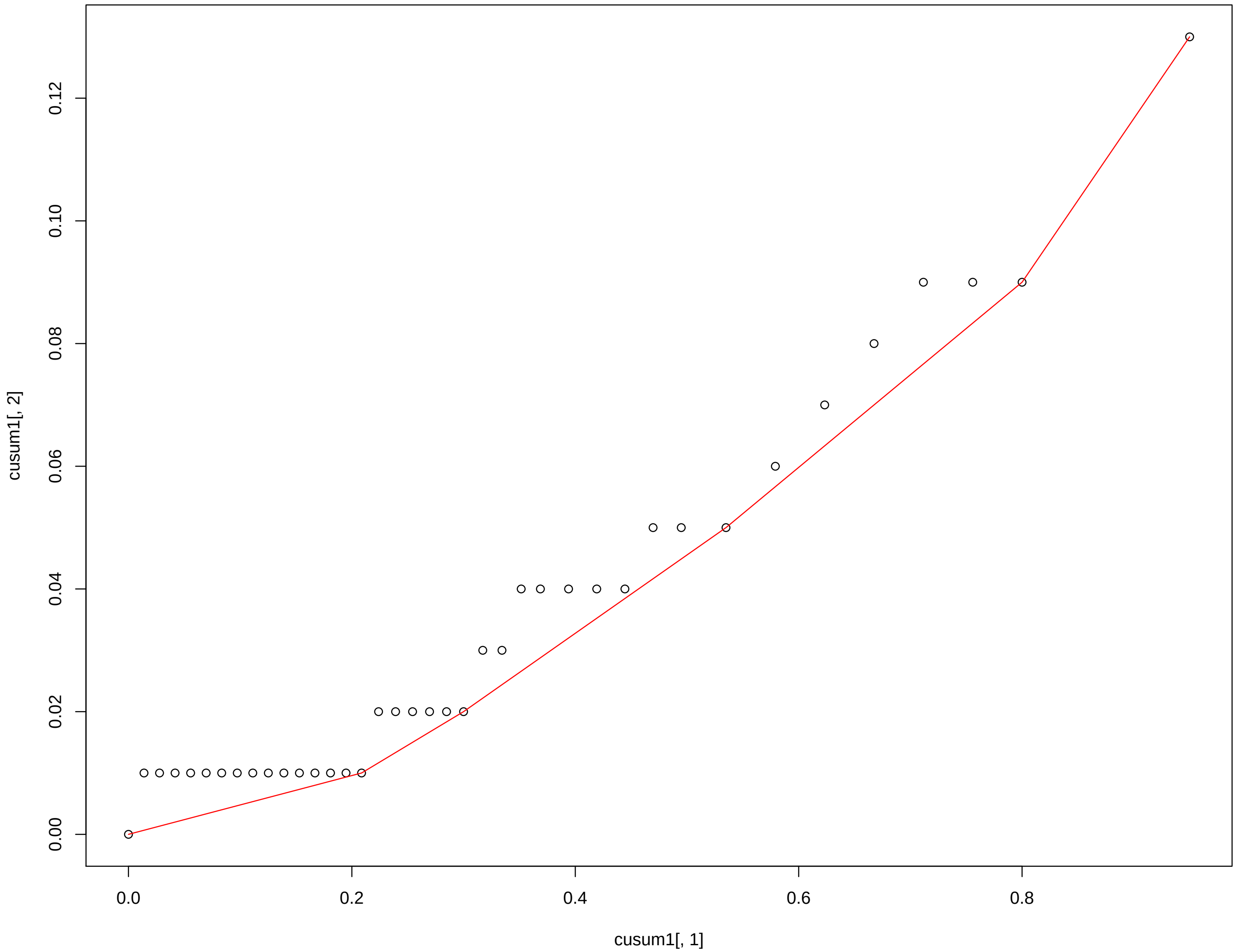

Figure 3: **Cusum diagram** $\{(G_{\hat{F}_{n+}}(t); V_{n1}(t)); t \in \mathbb{R}\}$. Simulation for $n = 100; K = 2$. $F_{0k}(t) = (k{=}3)\{1 - e^{-kt}\}; T \sim \mathrm{Unif}(0; 1{:}5)$.

Some history of work on the local rate

End of 2004: The following fact was proved.

Lemma 2. *Let* $\hat{F}_n = (\hat{F}_{n1}; :::; \hat{F}_{nK})$ *be the MLE of* $F_0 = (F_{01}; :::; F_{0K})$, *and let* $\hat{F}_{n+} = \sum_{k=1}^{K} \hat{F}_{nk}$ *and, similarly,* $F_{0+} = \sum_{k=1}^{K} F_{0k}$. *Moreover, let, for a* $\in (0;1)$, $\left[t_0 - \ ; t_0 + \ \right]$ *be an interval on which the components* F_{0k} *have continuous derivatives staying away from zero. Then there exists for each* $" > 0$ *and* $M > 0$ *an* $M_1 > 0$ *so that*

$$\mathbb{P}\left\{ \sup_{t \in [-M;M]} n^{1=3} \left| \hat{F}_{n+}(t_0 + n^{-1=3} t) - F_{0+}(t_0) \right| > M_1 \right\} < "; \; k = 1; :::; K:$$

This is not enough! To get localization of the $\hat{F}_{nk}$, we need that for t outside a neighborhood of order $O(n^{-1=3})$ of t_0 we can replace $G_{\hat{F}_{n+}}(t)$ by

$$G_{F_{0+}}(t) \stackrel{\text{def}}{=} \mathbb{G}_n(t) + n^{-1} \sum_{i=1}^{n} \frac{F_{0+}(T_i) - \Delta_{i+}}{1 - F_{0+}(T_i)} 1_{(-\infty;t]}(T_i);$$

up to terms of order $o_p((t - t_0)^2)$ in the self-induced cusum diagram:

$$\mathcal{P}_{nk} = \left\{ (G_{\hat{F}_{n+}}(t); V_{nk}(t)); \; t \in \mathbb{R} \right\}; \; V_{nk}(t) = n^{-1} \sum_{i=1}^{n} \Delta_{ik} 1_{(-\infty;t]}(T_i); \quad (15)$$

removing the self-inducedness of the coordinate $G_{\hat{F}_{n+}}$.

Solution in September 2005: strengthening of Kim-Pollard-type lemma.

Lemma 3. (**Global to local lemma for $\hat{F}_{n+}$**) *Let $\hat{F}_n$ be the MLE and let, for a* $\in (0;1)$, $\left[t_0 - 2\sqrt{\ }; t_0 + 2\sqrt{\ }\right]$ *be an interval on which the components F_{0k} have continuous derivatives staying away from zero. Then, for all $t \in [t_0 - \ ; t_0 + \ ]$ we have:*

$$\int_{\{|u-t_0|<|t-t_0|\}} \frac{|\hat{F}_{n+}(u) - F_{0+}(u|}{1 - \hat{F}_{n+}(u)} \, d\mathbb{G}_n(u) = n^{-1/6} O_p\left(n^{-1/2} \vee |t - t_0|^{3/2}\right); \text{ uniformly in } t \in [t_0 - \ ; t_0 + \ ]:$$

Note: $n^{-1/6} O_p\left(n^{-1/2} \vee |t - t_0|^{3/2}\right) = O_p(n^{-2/3})$ if $|t - t_0| = O_p(n^{-1/3})$.

Corollary 1. (**Tightness of $n^{1/3}\{\hat{F}_n(t_0 + n^{-1/3}t) - F_0(t_0)\}$**) *Let the conditions of Lemma 3 be satisfied. Then:*

(i) (Replacement of $G_{\hat{F}_{n+}}$ by $G_{F_{0+}}$)

$$G_{\hat{F}_{n+}}(t) = G_{F_{0+}}(t) + n^{-1/6} O_p\left(n^{-1/2} \vee |t - t_0|^{3/2}\right); \quad \text{uniformly in } t \in [t_0 - \ ; t_0 + \ ]:$$

(ii) For each $'' > 0$ *and* $M > 0$ *there exists an* $M_1 > 0$ *so that*

$$\mathbb{P}\left\{\sup_{t \in [-M;M]} n^{1/3} \left|\hat{F}_{nk}(t_0 + n^{-1/3}t) - F_{0k}(t_0)\right| > M_1\right\} < ''; \; k = 1; \ldots; K:$$

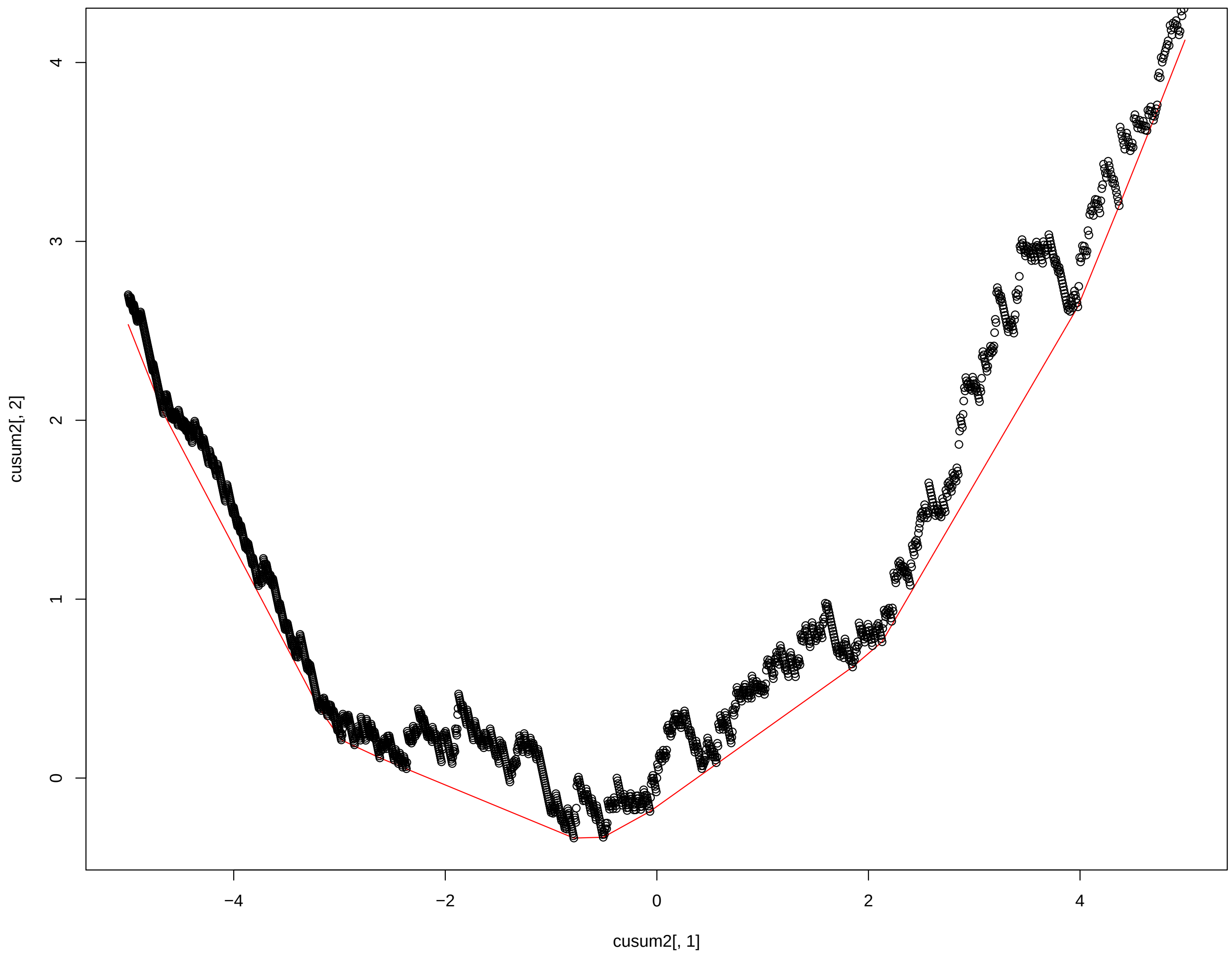

Figure 4: Localized cusum diagram $\left\{\left(n^{1/3}(G_{\hat{F}_{n+}}(t) - G_{\hat{F}_{n+}}(t_0)); n^{2/3}\left\{V_{n1}(t) - V_{n1}(t_0) - \int_{t_0}^{t} F_{01}(t_0)\, dG_{\hat{F}_{n+}}(u)\right\}\right); t \in \mathbb{R}\right\}$, $t_0 = 0{:}5; n = 10; 000$. Red curve: $n^{2/3}\int_{t_0}^{t}\{\hat{F}_{n1}(t_0) - F_{01}(u)\}\, dG_{\hat{F}_{n+}}(u)$. If $t < t_0$: $\int_{t_0}^{t}\{\hat{F}_{n1}(u) - F_{01}(t_0)\}\, dG_{\hat{F}_{n+}}(u) \stackrel{\text{def}}{=} -\int_{[t,t_0]}\{\hat{F}_{n1}(u) - F_{01}(t_0)\}\, dG_{\hat{F}_{n+}}(u)$.

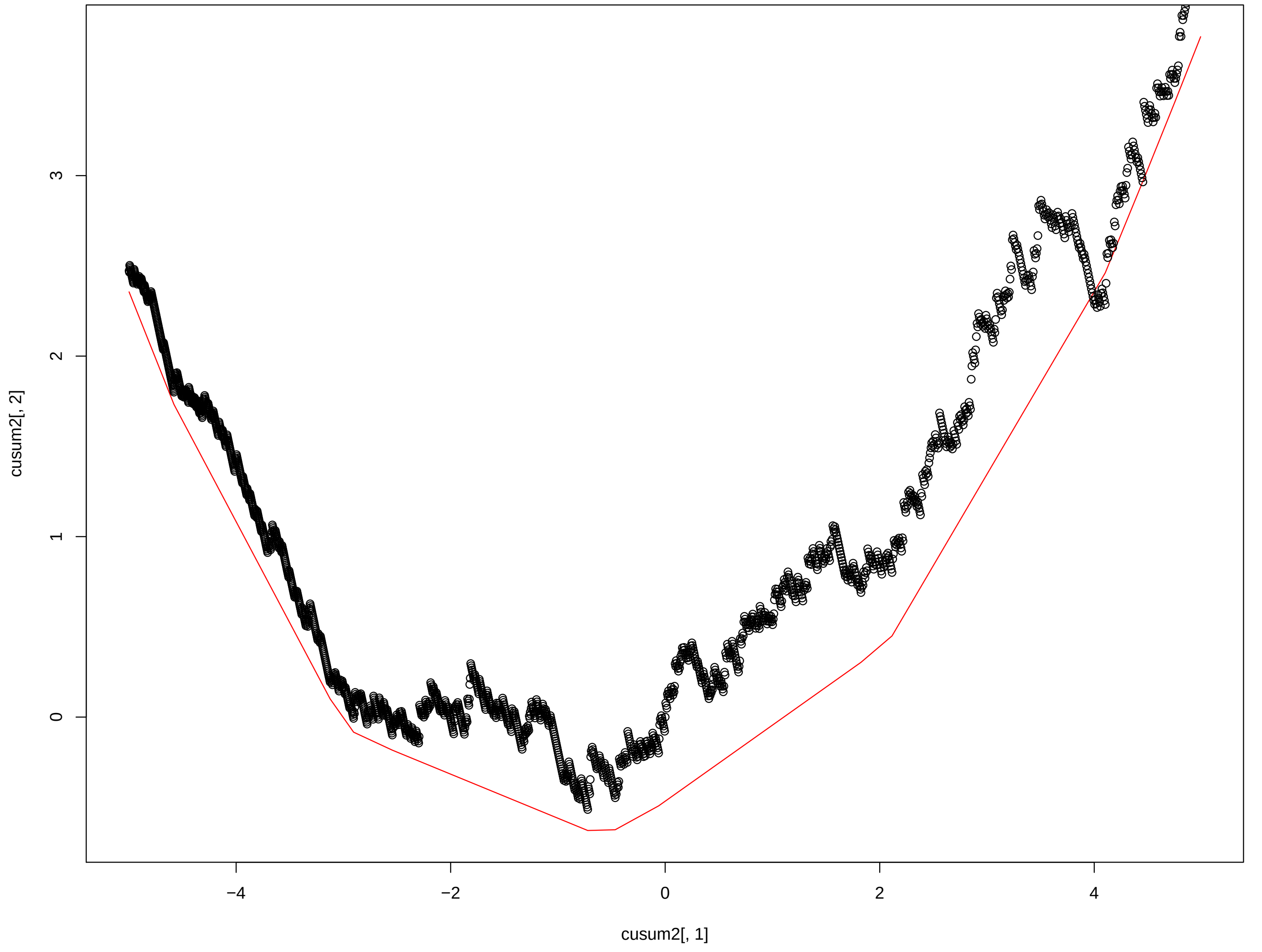

Figure 5: Localized cusum diagram with $G_{\hat{F}_{n+}}$ replaced by $G_{F_{0+}}$, $t_0 = 0.5, n = 10,000$. Red curve: $n^{2/3}\int_{t_0}^{t}\{\hat{F}_{n1}(u) - F_{01}(t_0)\}\,dG_{F_{0+}}(u)$. If $t < t_0$: $\int_{t_0}^{t}\{\hat{F}_{n1}(u) - F_{01}(t_0)\}\,dG_{F_{0+}}(u) \stackrel{\text{def}}{=} -\int_{[t,t_0]}\{\hat{F}_{n1}(u) - F_{01}(t_0)\}\,dG_{F_{0+}}(u)$.

Summary of the global to local argument

The MLE maximizes a global criterion. To extract the local limit behavior from this, we have to use some kind of characterization of the solution, for example a convex duality criterion.

1. In the case of simple current status data, this leads to a convex minorant characterization, which can be used for the determining the local behavior of the MLE.

2. In the case of competing risk with current status data, this leads to a self-induced convex minorant characterization, involving the sum $\hat{F}_{n+}$ of the individual MLE estimators $\hat{F}_{nk}$ for the several subdistribution functions F_{0k}. To get localization of the $\hat{F}_{nk}$, we need that for t outside a neighborhood of order $O(n^{-1/3})$ of t_0 we can replace $G_{\hat{F}_{n+}}(t)$ by

$$G_{F_{0+}}(t) \stackrel{\text{def}}{=} \mathbb{G}_n(t) + n^{-1}\sum_{i=1}^{n} \frac{F_{0+}(T_i) - \Delta_{i+}}{1 - F_{0+}(T_i)} 1_{(-\infty,t]}(T_i),$$

up to terms of order $o_p((t-t_0)^2)$ in the self-induced cusum diagram:

$$\mathcal{P}_{nk} = \left\{ \left(G_{\hat{F}_{n+}}(t), V_{nk}(t)\right),\ t \in \mathbb{R} \right\},\ V_{nk}(t) = n^{-1}\sum_{i=1}^{n} \Delta_{ik} 1_{(-\infty,t]}(T_i), \tag{16}$$

to get rid of the self-inducedness of the coordinate $G_{\hat{F}_{n+}}$ in the tightness argument. This is accomplished by the global to local lemma.

Limit distribution for competing risk

- First prove uniqueness of the limiting process, using tightness argument (Hardest part!)
- Localize characterization of limit process.
- Take subsequences of localized processes, based on a samples of size n, on $[-m; m]$. By tightness (using local rate result) there is a further subsequence that converges to some limit. Using a diagonal argument, it follows that there is a limit on $\mathbb{R}$. Here we go from local to global!
- By the continuous mapping theorem the limit must satisfy the limit characterization on $[-m; m]$ for each $m \in \mathbb{N}$.
- Letting $m \to \infty$ gives existence of the limiting process (almost for free!)
- By uniqueness of the limiting process, all subsequences converge to the same limit

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