Volatility Estimation with Functional Gradient Descent for Very High-Dimensional Financial Time Series

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Abstract

We propose a functional gradient descent algorithm (FGD) for estimating volatility and conditional covariances (given the past) for very high-dimensional financial time series of asset price returns. FGD is a kind of hybrid of nonparametric statistical function estimation and numerical optimization. Our FGD algorithm is computationally feasible in multivariate problems with dozens up to thousands of individual return series. Moreover, we demonstrate on some synthetic and real data-sets with dimensions up to 100, that it yields significantly, much better predictions than more classical approaches such as a constant conditional correlation GARCH-type model. Since our FGD algorithm is constructed from a generic algorithm, the technique can be adapted to other problems of learning in very high dimensions.

Heading: High-dimensional volatility estimation

1 Introduction

Returns of asset prices in a portfolio generate a multivariate time series, often in dozens or hundreds of dimensions. Denote by $P_{t,i}$ the price of asset i at time t and its returns by $X_{t,i} = (P_{t,i} - P_{t-1,i})/P_{t-1,i}$. The multivariate time series $\{X_{t,i}; t = 1, \ldots, n, i = 1, \ldots, d\}$ is usually assumed to be stationary and most of the modeling and prediction effort is on the multivariate (squared) volatility

$$V_t = \mathrm{Cov}_{d\times d}(\mathbf{X}_t|\mathcal{F}_{t-1}),\ \mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})^T, \tag{1.1}$$

where $\mathcal{F}_{t-1}$ denotes the information up to time $t-1$, the σ-algebra generated by $\{\mathbf{X}_s; s \leq t-1\}$. Besides the genuine interest of volatility in finance, V_t is a key quantity because the following model often yields a reasonable approximation,

$$\mathbf{X}_t = \Sigma_t \mathbf{Z}_t, \tag{1.2}$$

where $\Sigma_t \Sigma_t^T = V_t$ and $\mathbf{Z}_t$ are i.i.d. multivariate innovations with uncorrelated components and componentwise variances equal to one. Due to the enormous complexity of V_t as a function of the past $\mathcal{F}_{t-1}$ for d in the hundreds, the problem of predicting such high-dimensional V_t received only little attention, particularly when (nonlinear) methods are used which take cross-dependencies between the time series and auto-dependences from past values into account. Predicting truly high-dimensional volatility in (1.1) raises huge challenges in computational and modeling issues due to the well known curse of dimensionality. Previous work on multivariate volatility models has been given by Bollerslev (1990), Engle et al. (1990), Lin (1992) and Engle and Kroner (1995) in the framework of GARCH-type models, and by Harvey et al. (1994), Aguilar and West (2000) and Chib et al. (1999) within the stochastic volatility (state space model) framework. In the GARCH-type framework, only very simple models (Bollerslev, 1990) are feasible in high dimensions, whereas with stochastic volatility models, only Chib et al. (1999) present an example with dimensionality as large as 40 which is still far lower than the degree of multivariateness we can deal with here and which often occurs in practice.

We propose here a version of functional gradient descent (FGD), a recent technique from the area of machine learning (Breiman, 1999; Mason et al., 1999; Friedman et al., 2000; Friedman, 2001). Our FGD method is based on the likelihood-framework of a-priori very general GARCH-type models. FGD has mainly influenced the thinking around so-called Boosting (Freund and Schapire, 1996) which is a machine learning technique for the classification problem (predicting labels or classes from many explanatory variables). Some important modifications of more standard FGD algorithms are necessary to make the approach successful in the very different field here of high-dimensional financial time series.

It is well known that the multivariate approach is needed in many areas such as risk management and portfolio analysis. For example, the study of a portfolio

$$P_t = \sum_{i=1}^{d} \alpha_{t,i} P_{t,i}$$

with time-changing weights requires the multivariate approach. More generally, the aim is often to estimate the conditional distribution of a "pay-out" function

$$\psi_t(P_{t,1}, \ldots, P_{t,d})$$

given the information $\mathcal{F}_{t-1}$, where ψ_t could also be nonlinear. To do so, it suffices to estimate the distribution of $\mathbf{X}_t$ given $\mathcal{F}_{t-1}$; and this boils mainly down to estimating the volatility matrix V_t when assuming a model as in (1.2) (estimating the innovation distribution of $\mathbf{Z}_t$ is then easily done via estimated residuals).

Mainly for conceptual purposes, we present in section 3 FGD for volatility estimation in univariate time series. The method produces accurate results and is better than classical GARCH prediction. But as argued above, the merit is mainly for very high-dimensional problems where there is virtually no other competitive alternative method; this will be discussed in section 4. Numerical results are illustrated on real and simulated data.

2 The generic functional gradient descent algorithm

We are presenting here the main idea of functional gradient descent (FGD) in the framework of general regression. Consider data $(Y_1, X_1), \ldots, (Y_n, X_n)$, where Y_i is the response and X_i the predictor (explanatory) variable. For simplicity, we assume here that Y_i is univariate (e.g. taking values in $\mathbb{R}$) and X_i is p-dimensional (e.g. taking values in $\mathbb{R}^p$).

The aim is to estimate a function $F(x)$ such as $F(x) = \mathbb{E}[Y|X = x], F(x) = \mathrm{Var}(Y|X = x)$ or $F(x) = \mathbb{P}[Y = 1|X = x]$ if $Y \in \{0, 1\}$ is binary. The function $F(\cdot)$ can often be represented as the minimizer (in function space) of an expected loss function $\lambda(y, f)$,

$$F(\cdot) = \mathrm{argmin}_F \mathbb{E}[\lambda(Y, F(X))]$$

As an example, $F(x) = \mathbb{E}[Y|X = x]$ can be represented as the minimizer of such an expected loss with $\lambda(y, f) = |y - f|^2/2$. The functional gradient descent estimate of $F(\cdot)$ is then constructed from a constrained minimization of the empirical risk

$$n^{-1} \sum_{i=1}^{n} \lambda(Y_i, F(X_i)).$$

The constraints require that the solution $\hat{F}(\cdot)$ is an additive expansion of "simple estimates",

$$\hat{F}_M(\cdot) = \sum_{m=0}^{M} \hat{w}_m \hat{f}_m(\cdot). \tag{2.1}$$

The "simple estimates" are given from a statistical procedure $\mathcal{S}$, called the base learner, where

$$\mathcal{S}_X(U)(x) \text{ denotes the predicted value at } x \in \mathbb{R}^p \text{ from the base learner } \mathcal{S},$$

using the response vector $U = (U_1, \ldots, U_n) \in \mathbb{R}^n$ and predictor variables $X = (X_1, \ldots, X_n) \in \mathbb{R}^{pn}$. Typically, $\mathcal{S}_X(U)(x)$ is an estimate of $\mathbb{E}[U_1|X_1 = x]$. It is often constructed from (constrained or penalized) least squares fitting

$$\mathcal{S}_X(U)(\cdot) = \mathrm{argmin}_f \sum_{i=1}^{n} (U_i - f(X_i))^2.$$

It could be the fit from a base learner such as a regression tree, a projection pursuit or a neural net.

The constraints in the additive expansion in (2.1) are automatically built in when proceeding with the following generic FGD algorithm, cf. Friedman (2001).

Generic functional gradient descent (FGD)

Step 1 (initialization). Specify the starting function $\hat{F}_0(\cdot)$ and set $m = 1$.

Step 2 (projection of gradient to base learner). Compute the negative gradient

$$U_i = -\frac{\partial \lambda(Y_i, F)}{\partial F}|_{F=\hat{F}_{m-1}(X_i)}, \ i = 1, \ldots, n,$$

evaluated at the previous estimate $\hat{F}_{m-1}(\cdot)$ and the data points. Then, fit the negative gradient vector with a base learner $\mathcal{S}$

$$\hat{f}_m(\cdot) = \mathcal{S}_X(U)(\cdot).$$

The vector $(\hat{f}_m(X_1), \ldots, \hat{f}_m(X_n))^T$ can be viewed as a kind of projection of the negative gradient to the base learner.

Step 3 (line search). Perform a one-dimensional optimization for the step-length when up-dating $\hat{F}_{m-1}$ with $\hat{f}_m$,

$$\hat{w}_m = \text{argmin}_w \sum_{i=1}^{n} \lambda(Y_i, \hat{F}_{m-1}(X_i) + w\hat{f}_m(X_i)).$$

Up-date

$$\hat{F}_m(\cdot) = \hat{F}_{m-1}(\cdot) + \hat{w}_m \hat{f}_m(\cdot).$$

Step 4 (iteration and stopping). Increase m by one an iterate Steps 2 and 3 until stopping with $m = M$. This produces the FGD estimate

$$\hat{F}_M(\cdot) = \hat{F}_0(\cdot) + \sum_{m=1}^{M} \hat{w}_m \hat{f}_m(\cdot).$$

The stopping value M is chosen with the following cross-validation scheme: split the (in-sample) data into two sets, the first of size $0.7 \cdot n$ used as training set and the second of size $0.3 \cdot n$ as test set (this can also be used when the data are dependent). The optimal value of M is then chosen to optimize a cross-validated measure for test set prediction.

An instructive example is FGD with the quadratic loss function $\lambda(y, f) = |y - f|^2/2$. Then, the negative gradient in Step 2 becomes $U_i = Y_i - \hat{F}_{m-1}(X_i)$ $(i = 1, \ldots, n)$ which is the ordinary residual vector in iteration m. The estimated function $\hat{f}_m$ is the result of fitting the residuals $U_1, \ldots, U_n$ versus the predictor variables $X_1, \ldots, X_n$. The line search in Step 3 becomes trivial with $\hat{w}_m \equiv 1$, assuming that also $\hat{f}_m$ was fitted by least squares (possibly nonlinear or penalized). Therefore, FGD wit the quadratic loss function amounts

to iterative refitting of residuals and $\hat{F}_M(\cdot) = \hat{F}_0(\cdot) + \sum_{m=1}^{M} \hat{f}_m(\cdot)$, where each $\hat{f}_m(\cdot)$ is from fitting the current residuals versus X_i. For $M = 1$ (refitting the residuals once), the procedure has already been proposed by Tukey (1977) under the name "twicing". For this special case with the quadratic loss function and if the base learner is a linear projection (onto some known basis functions), the most simple example being the least squares fit in a linear model, FGD wouldn't do anything since the residual vector $(U_1, \ldots, U_n)$ is orthogonal to the projection space and $\hat{f}_m \equiv 0$ for all $m = 1, 2, \ldots$. But as soon as the learner is not a linear projection, FGD with the quadratic loss (and also with other loss functions) is an interesting way to estimate functions, particularly if the predictor space is high-dimensional, cf. Bühlmann and Yu (2001). FGD with a more general loss function may still be interpreted as iteratively fitting generalized residuals (negative gradient in Step 2) with a base learner.

Remark 1. Initialization in Step 1 was so far believed to be of negligible importance: an initial function often proposed is $\hat{F}_0 \equiv \overline{Y}_n$. But we will see in section 3, from an empirical point of view, that initialization does play an important role in the application of volatility estimation.

Remark 2. The line search in Step 3 guarantees that the empirical risk is monotonely decreasing with every iteration. It is exactly the same as used for finite-dimensional parameter optimization, cf. Nocedal and Wright (1999).

Remark 3. Stopping in Step 4 is important. Typically, the algorithm would fit the data perfectly as iterations tend to infinity, cf. Bühlmann and Yu (2001). Early stopping can be viewed as a regularization device which is very effective in complex model fitting. We find empirically that estimating M by the simple 70%-30% cross-validation scheme works well.

Numerical optimization has advanced to faster converging algorithms than steepest gradient methods. However, in our setting the slow "convergence" (note that we do not iterate until convergence) is very helpful in getting good regularization properties.

The name "functional gradient" suggests that we are doing gradient descent in function space. Contrary to a gradient descent scheme for a finite-dimensional parameter where the gradient is also of the same finite dimension, the negative gradient in Step 2 is an n-dimensional vector which can be interpreted as the values of an infinite-dimensional function (as n gets larger, the gradient-vector has higher dimension). This also indicates why we fit this n-dimensional negative gradient-vector by a base learner in order to have a more smooth approximation which is better in terms of predictive power (analogous to curve estimation by some sort of "smoothing"). FGD can be formulated in terms of infinite-dimensional function spaces: we are searching for the "direction" f (in function space) such that $\Lambda(\hat{F}_{m-1} + \varepsilon f)$ most rapidly decreases, for small value of ε, where $\Lambda(F) = n^{-1} \sum_{i=1}^{n} \lambda(Y_i, F(X_i))$. Viewing Λ as a functional on $\mathrm{lin}(\mathcal{F})$, the set of linear combinations of functions in a suitable class of base learners $\mathcal{F}$, the desired direction is the negative functional derivative $-d\Lambda(F, \cdot)$, where

$$d\Lambda(F, x) = \lim_{\varepsilon \searrow 0} \frac{\Lambda(F + \varepsilon \mathbb{I}_{[x]}) - \Lambda(F)}{\varepsilon}, \ x \in \mathbb{R}^p,$$

where $\mathbb{I}_{[\cdot]}$ denotes the indicator function. We are restricted to choose $f \in \mathcal{F}$ and cannot choose f as $-d\Lambda(F, x)$. Instead, we search for an f minimizing $\| - d\Lambda(F) - f\|^2$,

where $\|g\|^2 = n^{-1}\sum_{i=1}^n g(X_i)^2$. This is equivalent to fit the negative gradient vector $(U_1, \ldots, U_n)^T$ in Step 2 with the base learner $\mathcal{S}$ fitted by least squares producing $\hat{f}_m \in \mathcal{F}$.

Why should we use FGD at all? In very high-dimensional settings, particularly in connection with tree-structured base learners (see section 2.1), it is a computationally feasible method aiming to improve the starting function. The algorithm is greedy, never adjusting any of the previously fitted terms, producing a sequence of estimated models or predictions

$$\hat{F}_0 \prec \hat{F}_1 \ldots \prec \hat{F}_m \prec \hat{F}_{m+1} \ldots, \tag{2.2}$$

where "$\prec$" denotes "less complex" (fewer estimated parameters involved). Thus, FGD traces out a sequence of estimated predictions, and it is feasible to optimize such a one-dimensional sequence via choosing a stopping value M. The more classical approach would be to consider a set of models Γ, then fit every model $G \in \Gamma$ and finally select the model G_{opt} which optimizes a model-fitting criterion such as Akaike's information criterion (AIC). This approach often becomes computationally intractable in high dimensions (unless the model class Γ is very simple). For example, alternatively to our approach in section 4, we may wish to fit a multivariate GARCH model, more specifically a BEKK(1,1) model (Engle and Kroner, 1995) with $d = 10$ individual series. Very many of the hundreds of parameters would have to be set to zero in order to avoid overfitting; but this becomes an intractable model-selection problem with more than 10^{73} models to fit and check (when using a classical strategy for selecting the best subset of non-zero parameters with say the AIC criterion). Our approach with d up to $100 - 1000$ goes much beyond the order of magnitude of $d = 10$. The big question is of course whether a one-dimensional sequence of predictions as in (2.2) is good. Answering such a question from a theoretical point of view in general is very difficult and not developed so far. In a more simple setting, Bühlmann and Yu (2001) prove an asymptotic rate-optimality result for FGD with the quadratic loss function λ in one-dimensional function estimation of unknown smoothness. Thus, in such low-dimensional toy problems, FGD achieves some optimality criterion asymptotically. Good empirical performance of FGD, particularly in the area of classification with boosting (Freund and Schapire, 1996; Friedman et al., 2000; Friedman, 2001), and the asymptotic results in univariate function estimation mentioned above support evidence that FGD is a good strategy. These arguments are further strengthened by the fact that there are not many other methods whose computational costs remains manageable in very high-dimensional, large scale problems.

2.1 Choice of the base learner

The base learner in Step 2 of the functional gradient descent algorithm, producing the additive terms $\hat{f}_m(\cdot)$, obviously determines the FGD estimate $\hat{F}_M(\cdot)$. The base learner should be "weak", i.e. not too complex (not involving too many parameters to be estimated), so that FGD would not immediately produce an overfitted estimate in the first iteration. By adding further additive terms with every iteration, we increase complexity or dimensionality of the FGD estimate $\hat{F}_M$. However, this increase in complexity is not linear and performing further iterations typically changes complexity only by "small" amounts; for example, it is shown for certain cases to become exponentially diminishing as FGD iterations grow (Bühlmann and Yu, 2001). As usual, a bias-complexity trade-off is

present: the complexity of the base learner and the number of FGD iterations determine the trade-off in a somewhat unusual way, see also Bühlmann and Yu (2001).

Often, decision trees are used as base learners. Particularly in high dimensions, they have the ability to do variable selection by choosing just a few of the explanatory variables for prediction. We will consider here decision trees but also projection pursuit (Friedman and Stuetzle, 1981) as base learners which are both nonlinear. The choice of decision trees or projection pursuit learners should not be regarded as exclusive: others could be tried out and compared using some form of cross-validation. It is often desirable to make a base learner sufficiently "weak", i.e. of sufficiently low complexity. A simple but effective solution to achieve this is via shrinkage towards zero: the up-date $\hat{w}_m \hat{f}_m(\cdot)$ in Step 3 of the FGD algorithms is then replaced by

$$\nu \hat{w}_m \hat{f}_m(\cdot), \ 0 < \nu \leq 1. \tag{2.3}$$

Obviously, this reduces the variance (a complexity measure) by the factor ν^2.

3 Univariate volatility estimation with FGD

Having a univariate time series of observed prices $P_0, P_1, \ldots, P_n$ of an asset, we consider their returns $X_t = (P_t - P_{t-1})/P_{t-1}$, or $X_t = \log(P_t/P_{t-1})$ alternatively, and assume stationarity at least in a suitable time-window. The equi-distant time-spacing $\Delta t = t - (t-1) = 1$ is often one business day. Our working model for the time series of such price returns is

$$X_t = \sigma_t Z_t, \ \sigma_t^2 = F(X_{t-1}, X_{t-2}, \ldots), \tag{3.1}$$

where the innovations Z_t are i.i.d. with $\mathbb{E}[Z_t] = 0$, $\mathrm{Var}(Z_t) = 1$ and Z_t independent from $\{X_s; s < t\}$; the function $F : \mathbb{R}^\infty \to \mathbb{R}^+$ is assumed a priori to be very general. It is later the FGD estimate which constrains $F(\cdot)$ to be of a more particular, but often still rather general form.

For FGD, we assume that $F : \mathbb{R}^p \to \mathbb{R}^+$ with p finite; but we allow the starting function $\hat{F}_0(\cdot)$ to depend on the whole past of the time series. We choose the loss-function $\lambda(\cdot, \cdot)$ from the maximum-likelihood framework with innovations $Z_t \sim \mathcal{N}(0, 1)$:

$$\lambda(y, f) = -\log(f^{-1/2}\varphi(yf^{-1/2})) = \frac{1}{2}\left(\log(f) + \frac{y^2}{f} + \log(2\pi)\right).$$

(Of course, we could drop the $\log(2\pi)$ term which would then represent a simpler, equivalent loss function). This, because with Gaussian innovations in (3.1), the negative log-likelihood (conditional on the first p values) is

$$-\sum_{t=p+1}^{n} \log\left(F(X_{t-p}^{t-1})^{-1/2}\varphi(X_t F(X_{t-p}^{t-1})^{-1/2})\right),$$

where $X_{t-p}^{t-1} = X_{t-1}, \ldots, X_{t-p}$. The partial derivative of the loss-function is

$$\frac{\partial \lambda(y, f)}{\partial f} = (f^{-1} - y^2 f^{-2})/2$$

and the FGD algorithm from section 2 can now be used.

As a starting function, we propose to use the fit from a GARCH(1,1) model (Bollerslev, 1986)

$$\hat{F}_0(X_{t-1}, X_{t-2}, \ldots) = \hat{\alpha}_0 + \hat{\alpha}_1 X_{t-1}^2 + \hat{\beta}\hat{F}_0(X_{t-2}, X_{t-3}, \ldots), \tag{3.2}$$

with parameters estimates $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}$ from parametric maximum-likelihood in the GARCH(1,1) model with Gaussian innovations.

Summarizing, the FGD algorithm for univariate volatility estimation then looks as follows.

FGD for univariate volatility

Step 1 (initialization). Choose the starting function $\hat{F}_0(\cdot)$ from (3.2) and denote by $\hat{F}_0(t) = \hat{F}_0(X_1^{t-1})$. Set $m = 1$.

Step 2 (projection of gradient to base learner). Compute the negative gradient

$$U_t = (X_t^2 \hat{F}_{m-1}(t)^{-2} - \hat{F}_{m-1}(t)^{-1})/2, \ t = p+1, \ldots, n$$

Then, fit the negative gradient vector with a base learner, using always the first p time-lagged predictor variables (i.e. X_{t-p}^{t-1} is the predictor for U_t)

$$\hat{f}_m(\cdot) = \mathcal{S}_X(U)(\cdot).$$

Steps 3 and 4. As in the generic FGD algorithm, generally with shrinkage as in (2.3) and with stopping value M which optimizes the cross-validated log-likelihood.

The important issue is that the starting function $\hat{F}_0(\cdot)$ matters a lot for obtaining good volatility estimates. We illustrate this in Figure 3.1 by one simulation from the model (3.3) with sample size $n = 1000$. The out-sample OS-L_2 losses with $n_{out} = 1000$ (see section 3.1 below) are 174.5 (FGD with tree as in the Appendix and constant starting function $\hat{F}_0(x) \equiv S^2$ being the empirical overall variance), 95.5 (classical GARCH(1,1) prediction) and 70.5 (FGD with tree as in the Appendix and starting function from a GARCH(1,1) fit). We also compare the results from our optimal FGD algorithm with a semi-nonparametric (SNP) model (Gallant and Tauchen, 1989) which yielded 94.2 for the OS-L_2 loss. Similar results can also be obtained for real data, see section 3.2.

Also, it is advisable to allow for shrinkage in Step 3. Regarding the base learner $\mathcal{S}$, we have considered regression trees (Breiman at al., 1984) and projection pursuit regression (Friedman and Stuetzle, 1981). With regression trees, the FGD algorithm for univariate volatility estimation can be further modified to achieve additional marginal improvements: the version is described in the Appendix.

3.1 Numerical results for simulated data

We simulate from the following model,

$$\begin{aligned} &X_t = \sigma_t Z_t, \ \sigma_t^2 = F(X_{t-1}, \sigma_{t-1}^2), \\ &F(x, \sigma^2) = (0.1 + 0.2\,|x| + 0.9x^2) \cdot (0.8\exp(-1.5\,|x|\,|\sigma|)) + (0.4x^2 + 0.5\sigma^2)^{3/4}, \end{aligned} \tag{3.3}$$

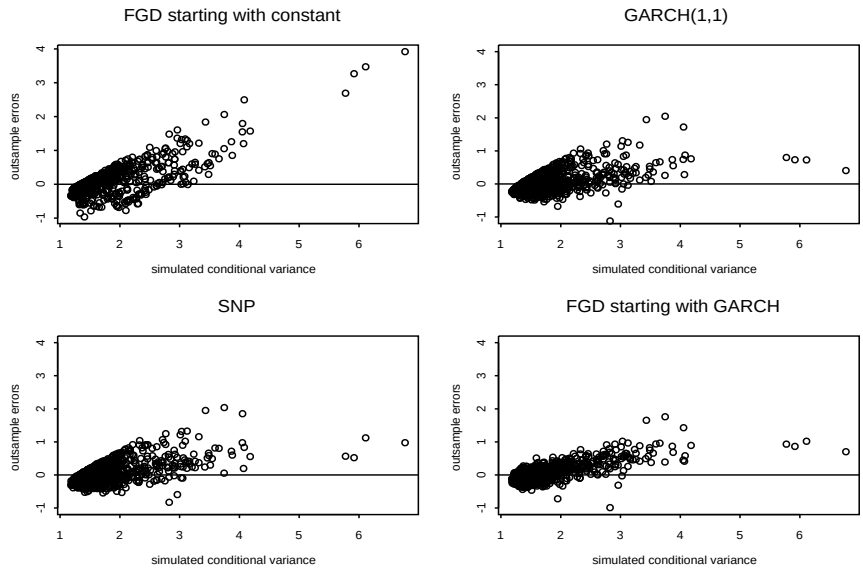

Figure 3.1: Outsample errors $\hat{\sigma}_t^2 - \sigma_t^2$ versus true σ_t^2 in simulated model (3.3). Top left: FDG with trees using constant starting function $\hat{F}_0(x) \equiv S^2$ (empirical marginal variance); Top right: GARCH(1,1) prediction; Bottom left: SNP prediction; Bottom right: FGD with trees using the GARCH(1,1) fit as starting function.

where $Z_t \sim \mathcal{N}(0,1)$ is as in (3.1).

For quantifying the goodness of fit, we consider various measures:

$$\text{out-sample negative log-likelihood: } \sum_{t=1}^{n_{out}} \lambda(Y_t, \hat{F}(Y_1^{t-1})), \tag{3.4}$$

$$\text{IS-L}_2 = \sum_{t=1}^{n} |\sigma_t^2 - \hat{F}(X_1^{t-1})|^2 \text{ (in-sample loss)}, \tag{3.5}$$

$$\text{OS-L}_2 = \sum_{t=1}^{n_{out}} |\sigma_t^2 - \hat{F}(Y_1^{t-1})|^2 \text{ (out-sample loss)}, \tag{3.6}$$

where $Y_1, \ldots, Y_{n_{out}}$ are new test observations, independent from but with the same distribution as the data $X_1, \ldots, X_n$; the estimated function $\hat{F}(\cdot)$ is based on the training data $X_1, \ldots, X_n$ only. Both, the IS- and OS-L_2 statistics are interesting measures in simulations, but we can't evaluate them for real data. The out-sample negative log-likelihood is a more generally applicable measure for out-sample performance. Table 3.1 shows the result for 50 independent realizations from model (3.3). Sample size is $n = 1000$ and test-set size is $n_{out} = 1000$. For this case, FGD with trees as in the Appendix is better than FGD with projection pursuit, and both FGD techniques outperform the predictions from classical GARCH(1,1) and SNP models. The differences in the out-sample log-likelihood are small despite that the actual differences in volatility are substantial. This phenomenon is well known and occurs because the out-sample log-likelihood measures quality for predicting *future returns* and not future volatilities; the former is much more noisy than the latter.

Model	Performance measure		
	OS $-$log-likelihood	IS-L_2	OS-L_2
GARCH(1,1)	1656.363	119.169	111.478
SNP	1659.077	128.9728	122.059
FGD with tree using $p = 1,\ \nu = 0.1,\ L = 3$	1654.361	96.4853	89.5336
FGD with tree using $p = 2,\ \nu = 0.1,\ L = 5$	1654.541	100.8058	93.2175
FGD with PPR using $p = 2,\ \nu = 0.05,\ S = 2$	1656.178	117.686	109.755

Table 3.1: Goodness of fit measures (on average) for fifty simulations from model (3.3) with sample size $n = 1000$. Out-sample performances OS-L_2 and negative log-likelihood as in (3.4)-(3.6) are evaluated with test-set of size $n_{out} = 1000$. Notation: number of lagged values (p), shrinkage factor (ν), number of terminal nodes (L) and number of ridge functions (S).

Thus, similar out-sample log-likelihoods (or other prediction losses) with different methods are not implying that the methods are similar in terms of the differences between estimated and true volatility, as observed in Table 3.1.

3.2 One real data example

We consider now negative daily log-returns of the DJIA index during the period December 22, 1993 until November 24, 1999 (1500 days). We obtain the following values for the out-sample negative log-likelihood (3.4), where the first 1000 observations were used for estimation and the remaining 500 served as a test set: 758.5 (FGD with tree as in the Appendix and (not advocated) constant starting function $\hat{F}_0(x) \equiv S^2$ being the empirical overall variance), 750.1 (classical GARCH(1,1) prediction), 749.5 (SNP prediction) and 746.4 (FGD with tree as in the Appendix and starting function from a GARCH(1,1) fit). Clearly for real data we can not evaluate the out-sample OS-L_2 losses and it is harder to detect differences for volatility prediction between the models (see the discussion at the end of section 3.1). We will show in section 4.3.2 how testing can be used to judge whether out-sample log-likelihoods (or other losses) are significantly different, even when they appear to be similar.

We also investigate here the functional behavior of the estimated volatility from our FGD method in comparison with the one from a GARCH(1,1) model. The estimated conditional variances and the news impact curve (Engle and Ng, 1993) for a GARCH(1,1) and a FGD fit are illustrated in Figure 3.2. We find that the differences are not big for this data set. However, we also see (for this example only to a minor extent), that the FGD method is capable to model asymmetry, a feature which can be very important for other data. Thus, our FGD method with regression trees as base learners allows good news and bad news to have a different impact on volatility, while the standard GARCH model does not; of course, other models such as EGARCH (Nelson, 1991) would also allow for asymmetries. Starting with a symmetric, centered around zero, news impact

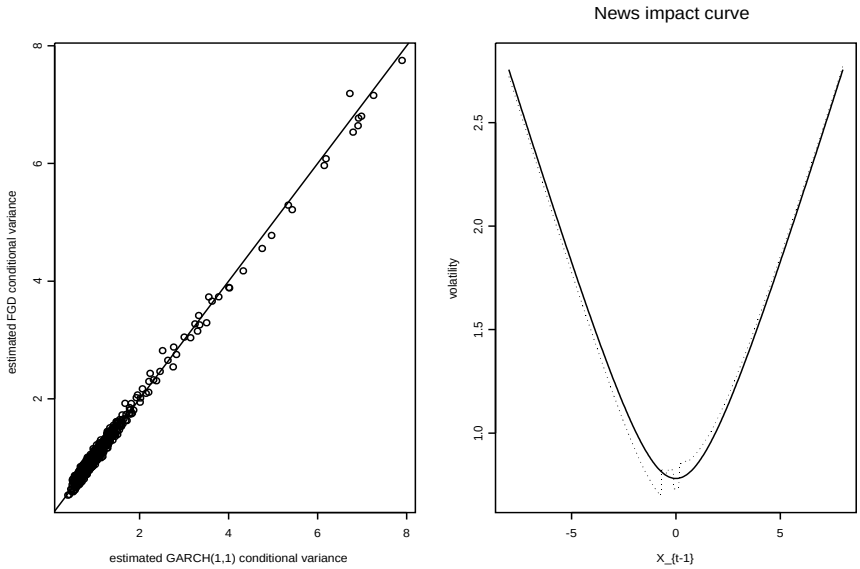

Figure 3.2: Left: estimated FGD conditional variances (with regression trees as base learners) versus estimated GARCH(1,1) conditional variances for a test-set of $n_{out} = 500$ daily negative log-returns of DJIA index. Right: news impact curve for the GARCH(1,1) model (solid line) and for the optimal FGD model with trees (dotted line) evaluated at the level of the unconditional variance of the stock return.

curve (from GARCH(1,1)), we build asymmetries adding constant parameters (different for every terminal node in the regression tree) as described by the algorithm in the Appendix. In the particular example shown in Figure 3.2, we see that, as we expect, bad news have slightly more impact on volatility in our FGD method than in the GARCH(1,1) model. Exactly the contrary happens if we consider good news (we remind here the reader that we consider negative log-returns).

4 Volatility estimation for high multivariate time series

In the multivariate set-up, we have time series of asset prices $\{P_{t,i}; t = 0, 1, \ldots, n, i = 1, \ldots, d\}$. Their returns are defined as

$$X_{t,i} = (P_{t,i} - P_{t-1,i})/P_{t-1,i},\ t = 1, \ldots, n.$$

As mentioned already in section 1, the challenging problem is prediction of the multivariate volatility matrix

$$V_t = \mathrm{Cov}_{d\times d}(\mathbf{X}_t|\mathcal{F}_{t-1}),\ \mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})^T$$

in dimensions in the hundreds. FGD becomes a powerful strategy to construct computable and good predictions for V_t.

We assume stationarity (at least within a suitable time-window). Our working model is a generalization of the constant conditional correlation (CCC) GARCH model (Bollerslev, 1990),

$$\mathbf{X}_t = \Sigma_t \mathbf{Z}_t, \tag{4.1}$$

where we assume the following:

(A1) (innovations) $\{\mathbf{Z}_t\}_{t\in\mathbb{Z}}$ is a sequence of i.i.d. multivariate innovations with spherical distribution (e.g. multivariate normal) having mean zero and covariance matrix $\mathrm{Cov}(\mathbf{Z}_t) = I_d$. Moreover, $\mathbf{Z}_t$ is independent from $\mathcal{F}_{t-1} = \{\mathbf{X}_s; s \le t-1\}$.

(A2) (CCC construction) The conditional covariance matrix $V_t = \mathrm{Cov}(\mathbf{X}_t|\mathcal{F}_{t-1}) = \Sigma_t\Sigma_t^T$ is almost surely positive definite for all t. The typical element of V_t is $v_{t,ij} = \rho_{ij}(v_{t,ii}v_{t,jj})^{1/2}$ $(i,j = 1,\ldots,d)$. The parameter $\rho_{ij} = \mathrm{Corr}(X_{t,i}, X_{t,j}|\mathcal{F}_{t-1})$ equals the constant conditional correlation and hence $-1 \le \rho_{ij} \le 1$, $\rho_{ii} = 1$.

(A3) (functional form) The conditional variances are of the form

$$v_{t,ii} = \sigma_{t,i}^2 = \mathrm{Var}(X_{t,i}|\mathcal{F}_{t-1}) = F_i(\{X_{t-j,k};\ j = 1,2,\ldots, k = 1,\ldots,d\})$$

where F_i takes values in $\mathbb{R}^+$.

Note that (A2) can be represented in matrix form as

$$\begin{aligned} &V_t = \Sigma_t\Sigma_t^T = D_t R D_t, \\ &D_t = \mathrm{diag}(\sigma_{t,1},\ldots,\sigma_{t,d}),\ R = [\rho_{ij}]_{i,j=1}^d. \end{aligned}$$

For estimating the functions $F_i(\cdot)$ in (A3), we propose FGD and restrict $F_i(\cdot) : \mathbb{R}^{pd} \to \mathbb{R}^+$ with p finite, i.e. involving the first p lagged multivariate observations. Estimation of the correlations can be easily done via empirical moments of residuals.

To proceed with a FGD technique, we first specify a suitable loss function. Assuming multivariate normality of the innovations $\mathbf{Z}_t$, the negative log-likelihood (conditional on the first p variables) is

$$\begin{aligned} &-\sum_{t=p+1}^{n} \log\left((2\pi)^{-d/2}\det(V_t)^{-1/2}\exp(-\mathbf{X}_t^T V_t^{-1}\mathbf{X}_t/2)\right) \\ =\ &\sum_{t=p+1}^{n}\left(\log(\det(D_t)) + \frac{1}{2}(D_t^{-1}\mathbf{X}_t)^T R^{-1}(D_t^{-1}\mathbf{X}_t)\right) + n' d\log(2\pi)/2 + n'\log(\det(R))/2 \end{aligned}$$

where D_t is diagonal with elements $\sqrt{F_i(\mathbf{X}_{t-p}^{t-1})}$ and $n' = n - p$. This motivates the following loss function

$$\begin{aligned} &\lambda_R(\mathbf{Y},\mathbf{f}) = \log(\det(D(\mathbf{f})) + \frac{1}{2}(D(\mathbf{f})^{-1}\mathbf{Y})^T R^{-1}(D(\mathbf{f})^{-1}\mathbf{Y}) + \frac{1}{2}\log(\det(R)) + \frac{d}{2}\log(2\pi), \\ &D(\mathbf{f}) = \mathrm{diag}(f_1,\ldots,f_d). \end{aligned} \tag{4.2}$$

(The terms $d\log(2\pi)/2$ and $\log(\det(R))/2$ are constants and could be dropped). As pointed out with the subscript, the loss function depends on the unknown correlation matrix R.

Our FGD algorithm will be constructed iteratively by estimating R and using the loss function with the estimated R to get an estimate for all F_i's.

Having a (previous) estimate $\hat{\mathbf{F}} = (\hat{F}_1, \ldots, \hat{F}_d)$, we then construct the following estimate for the correlation matrix R. Build the residuals

$$\hat{\varepsilon}_{t,i} = X_{t,i}/\hat{F}_i(\mathbf{X}_{t-1}, \ldots)^{1/2},\ t = p+1, \ldots, n$$

and define

$$\hat{R} = (n-p)^{-1} \sum_{t=p+1}^{n} \hat{\varepsilon}_t \hat{\varepsilon}_t^T,\ \hat{\varepsilon}_t = (\hat{\varepsilon}_{t,1}, \ldots, \hat{\varepsilon}_{t,d})^T. \tag{4.3}$$

The partial derivatives of the loss function are

$$\frac{\partial \lambda_R(\mathbf{Y}, \mathbf{f})}{\partial f_i} = (f_i - \sum_{j=1}^{d} \frac{\gamma_{ij} y_i y_j}{f_i^{3/2} f_j^{1/2}})/2,\ i = 1, \ldots, d, \tag{4.4}$$

where $[\gamma_{ij}]_{i,j=1}^d = R^{-1}$. This will be used when computing negative gradients (see Step 2 in the generic FGD algorithm) for every component $i = 1, \ldots, d$.

As a starting function, we propose to use the fit from a CCC-GARCH(1,1) model (Bollerslev, 1990) which is of the form (4.1) with (A3) specified to

$$F_i(X_{t-1}, X_{t-2}, \ldots) = \sigma_{t,i}^2 = \alpha_{0,i} + \alpha_{1,i} X_{t-1,i}^2 + \beta_{0,i} \sigma_{t-1,i}^2,\ i = 1, \ldots, d. \tag{4.5}$$

For d large, the estimates are constructed with maximum likelihood from the d individual series. This ignores the more general correlation structure in R, causing some statistical decrease in efficiency, but gaining the advantage that the individual estimates are computable (in parallel) in very high dimensions d.

The FGD algorithm for multivariate volatility looks as follows.

FGD for multivariate volatility

Step 1 (initialization). Choose the starting function $\hat{F}_{i,0}(\cdot)$ and denote by $\hat{F}_{i,0}(t) = \hat{F}_{i,0}(\mathbf{X}_{t-1}, \mathbf{X}_{t-2}, \ldots)$ $(i = 1, \ldots, d)$. Compute $\hat{R}_0$ as in (4.3) using $\hat{\mathbf{F}}_0$. Set $m = 1$.

For every component $i = 1, \ldots, d$, do the following.

Step 2$_i$ (projection of component gradients to base learner). Compute the negative gradient

$$U_{t,i} = -\frac{\partial \lambda_{\hat{R}_{m-1}}(\mathbf{X}_t, \mathbf{F})}{\partial F_i}|_{\mathbf{F} = \hat{\mathbf{F}}_{m-1}(t)},\ t = p+1, \ldots, n.$$

This is explicitly given in (4.4). Then, fit the negative gradient vector $U_i = (U_{p+1,i}, \ldots, U_{n,i})^T$ with a base learner, using always the first p time-lagged predictor variables (i.e. $\mathbf{X}_{t-p}^{t-1}$ is the predictor for $U_{t,i}$)

$$\hat{f}_{m,i}(\cdot) = \mathcal{S}_X(U_i)(\cdot).$$

Step 3_i *(line search).* Perform one-dimensional optimization for the step-length,

$$\hat{w}_{m,i} = \text{argmin} \sum_{t=p+1}^{n} \lambda_{\hat{R}_{m-1}}(\mathbf{X}_t, \hat{\mathbf{F}}_{m-1}(t) + w\hat{f}_{m,i}(\mathbf{X}_{t-p}^{t-1})).$$

($\hat{\mathbf{F}}_{m-1}(t) + w\hat{f}_{m,i}(\cdot)$ is defined as the function which is constructed by adding in the ith component only). This can be expressed more explicitly by using (4.2).

Step 4 (up-date). Select the best component as

$$i_m^* = \text{argmin}_i \sum_{t=p+1}^{n} \lambda_{\hat{R}_{m-1}}(\mathbf{X}_t, \hat{\mathbf{F}}_{m-1}(t) + \hat{w}_{m,i}\hat{f}_{m,i}(\mathbf{X}_{t-p}^{t-1})).$$

Up-date

$$\hat{\mathbf{F}}_m(\cdot) = \hat{\mathbf{F}}_{m-1}(\cdot) + \hat{w}_{m,i_m^*}\hat{f}_{m,i_m^*}(\cdot).$$

Then, compute the new estimate $\hat{R}_m$ according to (4.3) using $\hat{\mathbf{F}}_m$.

Step 5 (iteration). Increase m by one and iterate Steps 2–4 until stopping with $m = M$. This produces the FGD estimate

$$\hat{\mathbf{F}}_M(\cdot) = \hat{\mathbf{F}}_0(\cdot) + \sum_{m=1}^{M} \hat{w}_{m,i_m^*}\hat{f}_{m,i_m^*}(\cdot).$$

As in the generic algorithm, the stopping value M is chosen to optimize the cross-validated log-likelihood.

Note that shrinkage as in (2.3) is often useful in Steps 2–4. As in the univariate case, the starting function $\hat{F}_0(\cdot)$ matters a lot for obtaining good volatility estimates.

A crucial difference to the multivariate (multi-class) FGD algorithm from Friedman et al. (2000), who propose to cycle through the dimensions in a systematic way one after the other, is that our construction is with candidate components in Steps 2_i and 3_i and choosing the component i^* in Step 4 which brings the most substantial improvement ("the steepest direction") in a single FGD iteration. Cycling through in a systematic way forces to add complexity of the FGD estimate for every component: but this isn't realistic if one time series is "of simpler structure" than others. We illustrate this for one realization of a 3-dimensional model as in (4.1): the volatilities are given by F_1 and F_2 from (4.8) below with fixed parameters $\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.2, \alpha_4 = 0.75, \alpha_5 = 0.5$ and F_3 from (4.9) below with fixed parameters $\alpha_1 = 0.1,\ \alpha_2 = 0.9,\ \alpha_3 = -1.5, \alpha_4 = 0.5$. Table 4.1 impressively demonstrates that choosing the best component i^* brings substantial improvements for the third series. The goodness of fit criteria used here are the following. The out-sample negative log-likelihood function (the out-sample loss λ) is, similarly to (3.4),

$$\sum_{t=n+1}^{n+n_{out}} \lambda_{\hat{R}}(\mathbf{X}_t, \hat{\mathbf{F}}(t)), \tag{4.6}$$

where the estimates denoted by a "$\hat{}$" are based on the training data $\mathbf{X}_1, \ldots, \mathbf{X}_n$. The other criteria are the univariate in- and out-sample L_2-losses from (3.5) and (3.6).

Model	Performance measure						
	OS $-$log-lik.	IS-L_2			OS-L_2		
		series$_1$	series$_2$	series$_3$	series$_1$	series$_2$	series$_3$
CCC-GARCH(1,1)	3268.6	207.38	207.02	193.84	147.55	173.68	200.35
FGD with trees using systematic cycling	3052.2	151.62	169.60	224.19	88.351	123.33	241.77
FGD with trees selecting best component i^*	3053.5	153.89	154.20	191.95	90.646	121.36	195.58

Table 4.1: Goodness of fit measures for one three-dimensional realization of size $n = 1000$ from model (4.1) with $\mathbf{Z}_t \sim \mathcal{N}_3(0, I)$ and individual conditional variances defined by (4.8) and (4.9). Out-sample negative log-likelihood as in (4.6) and individual OS-L_2 as in (3.6) are evaluated with test-set of size $n_{out} = 1000$.

4.1 Base learners with variable selection

Regarding the base learner $\mathcal{S}$, we have considered regression trees (Breiman at al., 1984). Fitting regression trees for the components in our multivariate FGD (Step 2_i) can be modified as in the univariate setting described in the Appendix. In very high dimensions, it is essential to use a base learner which selects only a few variables from a huge predictor space. Decision trees have this property: when having L terminal nodes, the decision tree base learner selects at most $L - 1$ different explanatory variables. In combination with FGD, we would then add to the starting function $\hat{F}_0$ an additive correction involving at most $M(L-1)$ different variables which may be much lower than dp which is the number of the p lagged predictor variables in every of the d time series.

4.2 Computational cost and parallelization

The computational complexity of our multivariate FGD algorithm (without initialization) is

$$Md \cdot \text{complexity}(\mathcal{S}) + Md \cdot \text{complexity}(\text{line search}) + M \cdot \text{complexity}(\hat{R})$$

The complexity of the estimate $\hat{R}$ in (4.3) is quadratic in the dimension d but the computational cost of this moment estimator is not substantial relative to the other tasks. The numerical line search has to be done Md times which can contribute substantially to computing time when d is in the hundreds, M up to 100 (which is most often big enough) and n around 1000 (which is at the upper range when using daily financial data, due to possible non-stationarity). Likewise, the base learner has to be fitted Md times. When d gets large, we encounter in addition that fitting of the base learner becomes typically more costly (complexity($\mathcal{S}$) also depends on d). For example with $\mathcal{S}$ a decision tree, the computational complexity of $\mathcal{S}$ grows linearly in d. However, for d in the range of 100, M up to 100 and n about 1000, the total complexity is still feasible when using decision trees.

When d is in the order of 1000, the simple implementation of the FGD algorithm becomes quickly computationally expensive. Fortunately, it is very easy to implement a parallel version. Searching the best component i_m^* in Steps 2_i and 3_i requires consideration of all components $i = 1, \ldots, d$: this can be parallelized immediately, reducing the computational cost a lot. Furthermore, when using trees as base learner in high dimensions, their fitting, which requires visiting all components $i = 1, \ldots, d$ could also be substantially parallelized. While this is a bit more sophisticated, parallelization of the Steps 2_i and 3_i is immediate. Such simple parallelizations make our FGD algorithm immediately feasible in dimensions d in the thousands.

4.3 Numerical results

4.3.1 100-dimensional simulated data

We simulate a 100-dimensional series of sample size $n = 1000$ from model (4.1) with $\mathbf{Z}_t \sim \mathcal{N}_{100}(0, I)$ and various volatility functions F_j. One such function is the classical GARCH(1,1) volatility

$$\begin{aligned}
&\sigma_{t,i}^2 = F_i(X_{t-1,i}, \sigma_{t-1,i}^2) \text{ where} \\
&F_i(x, \sigma^2) = \alpha_0 + \alpha_1 x^2 + \beta\sigma^2, \text{ where} \\
&\alpha_0 \sim \text{Unif}([0, 0.2]),\ \alpha_1 \sim \text{Unif}([0.05, 0.15]),\ \beta \sim \text{Unif}([0.8, 0.84])
\end{aligned} \tag{4.7}$$

and $\alpha_0, \alpha_1, \beta$ mutually independent. Another function is from a threshold model

$$\begin{aligned}
&\sigma_{t,i}^2 = F_i(X_{t-1,i}, \sigma_{t-1,i}^2) \text{ where} \\
&F_i(x, \sigma^2) = \begin{cases} \alpha_1 + \alpha_2 x^2\ , & \text{if } x \le d_1 = 0, \\ 0.2 + \alpha_3 x^2 + \alpha_4 \sigma^2\ , & \text{if } x > d_1 = 0 \text{ and } \sigma^2 \le d_2 = 0.5, \\ 0.8 + \alpha_5 \sigma^2\ , & \text{if } x > d_1 = 0 \text{ and } \sigma^2 > d_2 = 0.5, \end{cases} \quad \text{where} \\
&\alpha_1 \sim \text{Unif}([0, 0.3]),\ \alpha_2 \sim \text{Unif}([0.4, 0.6]),\ \alpha_3 \sim \text{Unif}([0.1, 0.3]), \\
&\alpha_4 \sim \text{Unif}([0.6, 0.8]),\ \alpha_5 \sim \text{Unif}([0.4, 0.6])
\end{aligned} \tag{4.8}$$

($\alpha_1, \ldots, \alpha_5$ mutually independent). A third and a fourth function, in which we also allow for one cross-terms, are

$$\begin{aligned}
&\sigma_{t,i}^2 = F_i(X_{t-1,i}, X_{t-1,j}, \sigma_{t-1,i}^2) \text{ where} \\
&F_i(x, y, \sigma^2) = (\alpha_1 + 0.2\,|y| + \alpha_2 x^2) \cdot (0.8 \exp(\alpha_3\,|x|\,|\sigma|)) + (0.4x^2 + \alpha_4\sigma^2)^{3/4}, \\
&\alpha_1 \sim \text{Unif}([0.05, 0.15]),\ \alpha_2 \sim \text{Unif}([0.8, 0.95]), \\
&\alpha_3 \sim \text{Unif}([-1.6, -1.4]),\ \alpha_4 \sim \text{Unif}([0.4, 0.6])
\end{aligned} \tag{4.9}$$

($\alpha_1, \ldots, \alpha_4$ mutually independent), and

$$\begin{aligned}
&\sigma_{t,i}^2 = F_i(X_{t-1,i}, X_{t-1,j}, \sigma_{t-1,i}^2) \text{ where} \\
&F_i(x, y, \sigma^2) = (0.1 + \alpha_1\,|y|^3) \cdot \exp(\alpha_2 x^2) + \alpha_3 (\sigma^2)^{3/4}, \\
&\alpha_1 \sim \text{Unif}([0.1, 0.2]),\ \alpha_2 \sim \text{Unif}([-0.1, 0]),\ \alpha_3 \sim \text{Unif}([0.8, 0.9])
\end{aligned} \tag{4.10}$$

$(\alpha_1, \alpha_2, \alpha_3$ mutually independent), where the component $j \in \{1, \ldots, d\} \setminus i$ is chosen randomly. Each of the volatility functions F_i is randomly chosen with probability 1/4, independent of each other, to generate 100 such functions in total. Note that also the coefficients in these functions are randomly chosen. The constant conditional correlation matrix R is chosen to mimic the one of real log-returns. This model is "fairly close" to a CCC-GARCH(1,1) model since half of the volatility functions involve only auto-dependence (no dependence on a cross-series in (4.7) and (4.8)), a quarter of them actually being linear GARCH-type, and the other half involve only one other cross-series.

The results are displayed in Table 4.2 and Figure 4.1 (test set size is $n_{out} = 1000$).

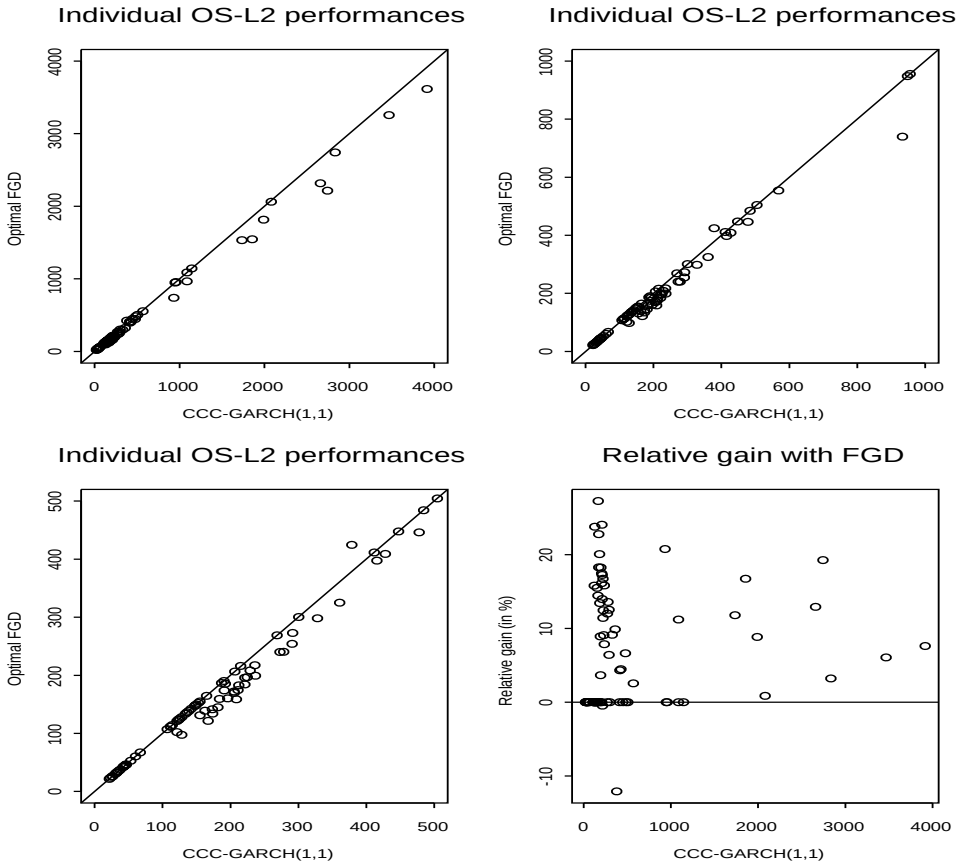

Figure 4.1: Individual OS-L_2 measures as in (3.6) for one 100-dimensional realization of size $n = 1000$ as described above. First three panels: performance of CCC-GARCH(1,1) on x-axis and of FGD with trees on y-axis with the line indicating equal performance, on different scales. Lower right panel: relative gains with FGD with trees of individual OS-L_2 measures (see 3.6) on y axis versus performance of CCC-GARCH(1,1) on x-axis.

We observe only a small gain of FGD with trees over the CCC-GARCH(1,1) prediction with respect to the negative log-likelihood, because the signal to noise level is low, but a more substantial gain when looking at individual outsample L_2-losses as in (3.6) where

Model	Performance measure			
	OS $-$log-lik.	ave(OS-$L_2^{(j)}$)	max gain in OS-L_2	min gain in OS-L_2
CCC-GARCH(1,1)	121636.8	447.7096	–	–
FGD with tree using $p = 2$, $\nu = 0.5,\ L = 5$	119884.8 (1.4%)	413.5867 (7.6%)	27.3%	-12.1%

Table 4.2: Goodness of fit measures for one 100-dimensional realization of size $n = 1000$ as described above. The measures are defined as in (4.6), as the average over all $d = 100$ individual series as in (3.6) and the extremal relative individual gains with respect to (3.6); test-set size is $n_{out} = 1000$ and relative gains with FGD over CCC-GARCH(1,1) are given in parentheses.

the observation noise is not present. Of course, we cannot expect to learn in all $d = 100$ components with sample size $n = 1000$ (or here in all 75 components which are not of linear GARCH(1,1)-type). On average, the OS-L_2 gain over all $d = 100$ components is 7.6%: this gain will generally decrease when dimension d increases and keeping sample size n fixed. However, we also see from Figure 4.1, that FGD mainly improves (in absolute terms) at those components where the CCC-GARCH(1,1) predictions are poorest. This is consistent with the intuition that functional gradient descent improves the "hardest" cases. Regarding the relative gains with FGD: they range from -12.1 to 27.3% (but only in two out of 100 cases there is a loss) and quite many large relative gains are realized where the CCC-GARCH predictions are fairly low (see lower right panel of Figure 4.1).

4.3.2 A seven-dimensional real data example

We consider also real daily return data from seven financial indices. The dimensionality $d = 7$ is "mid-range" allowing for a representation of the results which is easy to survey. FGD will also turn out to be useful in such orders of dimensions. The data comprises 1500 daily returns from the US DJIA, the French CAC40, the German DAX, the Italian BCI, the Dutch CBS, the British FTAS and the Japanese NIKKEI index, during the period January 31, 1990 until September 9, 1996: we use the first 1000 time points for training (fitting) and the remaining 500 for out-sample testing (evaluation).

We consider two goodness of fit measures for such real data. One is the negative out-sample log-likelihood in (4.6). Alternatively, we also look at the individual out-sample prediction L_2 losses,

$$\text{OS-PL}_2 = \sum_{t=1001}^{1500} |X_{t,i}^2 - \hat{F}_i(t)|^2 \ (i = 1, \ldots, d).$$

The IS-PL_2 is defined analogously using the training sample only. The results are given in Table 4.3.

With real data, differences in the out-sample negative log-likelihood or OS-PL_2 can be small between different methods; see also the discussion at the end of section 3.1. The

	Performance measure					
	CCC-GARCH(1,1)			FGD with trees		
	OS $-$log-lik.	IS-PL_2	OS-PL_2	OS $-$log-lik.	IS-PL_2	OS-PL_2
global	4369.81			4357.46		
DJIA		3127.25	451.817		3091.49	448.297
CAC40		17138.9	6781.32		17009.1	6734.14
DAX		19931.6	5826.86		19931.6	5826.86
BCI		25135.4	2942.45		25135.4	2942.45
CBS		7437.44	3087.85		7437.44	3087.85
FTAS		10714.2	777.266		10714.2	777.266
NIKKEI		61011.8	5345.68		60603.8	5303.75

Table 4.3: Goodness of fit measures for a seven-dimensional real data example. The optimal parameters in FGD with trees are $p = 3$ lagged values, shrinkage $\nu = 0.5$ and $L = 3$ terminal nodes.

out-sample criterion is here generally denoted by

$$\sum_{t=1}^{n_{out}} L_t,$$

where L_t is the out-sample loss for out-sample prediction at time t depending on a technique or model, i.e. $L_t = L_{t,\text{model}}$ (we first consider $L_t = \lambda_{\hat{R}}(\mathbf{X}_t, \hat{\mathbf{F}}(t))$, the multivariate negative out-sample log-likelihood). The differences between two models with respect to out-sample performance is then

$$\Delta(\text{model}_1, \text{model}_2) = \sum_{t=1}^{n_{out}} D_t, \;\; D_t = L_{t,\text{model}_1} - L_{t,\text{model}_2}.$$

We aim to test whether $\mathbb{E}[D_t] = 0$ against one- (or two-) sided alternatives. We consider versions of the t- and sign-test, adapted to the case of dependent observations D_t.

The t-type test statistic is

$$\begin{aligned} & n_{out}^{1/2} \frac{\overline{D}}{\hat{\sigma}_{D,\infty}}, \;\; \overline{D} = n_{out}^{-1} \sum_{t=1}^{T} D_t, \\ & \hat{\sigma}_{D,\infty}^2 = 2\pi \hat{f}_D(0) \end{aligned} \tag{4.11}$$

where $\hat{f}_D(0)$ is an estimate of $f_D(0) = (2\pi)^{-1} \sum_k Cov(D_0, D_k)$, the spectral density at frequency zero of the process $\{D_t\}_t$, where we assume that $\{D_t\}_t$ is stationary and suitably regular, so that

$$n_{out}^{1/2}(\overline{D} - \mathbb{E}[D_t]) \Rightarrow \mathcal{N}(0, \sigma_{D,\infty}^2)$$

t-type test	sign-type test
-0.88 (0.189)	-1.73 (0.042)

Table 4.4: Test-statistics and P-values (in parentheses) for one-sided testing for differences in multivariate out-sample negative log-likelihood of the FGD with trees and the CCC-GARCH predictions. Negative test-statistics favor the FGD method.

From this, we obtain asymptotically the standard normal distribution of the test-statistic in (4.11) under the null-hypothesis and hence the ingredients to perform the test.

The sign-type test statistic is

$$n_{out}^{1/2}\frac{\overline{W}-1/2}{\hat{\sigma}_{W,\infty}},\ \overline{W}=n_{out}^{-1}\sum_{t=1}^{T}\mathbb{I}_{[D_t>0]},$$
$$\hat{\sigma}_{D,\infty}^2=2\pi\hat{f}_D(0) \tag{4.12}$$

where $\hat{f}_W(0)$ is an estimate of the spectral density at frequency zero of the process $\{\mathbb{I}_{[D_t>0]}\}_t$, where we assume that $\{\mathbb{I}_{[D_t>0]}\}_t$ is stationary and suitably regular, so that the central limit theorem holds. This then implies that the test-statistic in (4.12) is asymptotically standard normal under the null-hypothesis $H_0 : p = \mathbb{P}[D_t > 0] = 1/2$ which allows to perform the test. For both tests, estimation of the spectral density at zero is done via smoothing the periodogram.

We apply both tests for comparing the FGD with trees and the CCC-GARCH predictions. The alternative is one-sided where FGD with trees has lower out-sample loss. The results are given in Table 4.4. Thus, the small difference of 0.3% in the out-sample log-likelihoods of FGD and CCC-GARCH(1,1) from Table 4.3 turns out to be significant at the 5%-level for the sign-type test.

Moreover, we see that our multivariate FGD algorithm with candidate components improves three series (DJIA, CAC40 and NIKKEI), leaving the estimates for the other four series unchanged. We can consider these three series individually and perform the same t-type and sign-type tests for the univariate out-sample negative log-likelihood given by (3.4). The results are summarized in Table 4.5.

	t-type test	sign-type test
DJIA	-0.39 (0.349)	-0.49 (0.311)
CAC40	-1.75 (0.040)	-2.26 (0.012)
NIKKEI	-1.71 (0.043)	-0.10 (0.460)

Table 4.5: Test-statistics and P-values (in parentheses) for one-sided testing for differences in individual univariate out-sample negative log-likelihood of the FGD with trees and the CCC-GARCH predictions. Negative test-statistics favor the FGD method.

On the level of individual series, mainly the CAC40 index return series is significantly improved by FGD modeling.

5 Conclusions

We have presented an FGD algorithm which is a technique for estimation of the conditional covariance matrix in (1.1). It is computationally feasible in multivariate problems with several hundreds up to thousands of return series, and regularization to protect against overfitting is simple. This is not the case for other multivariate methods such as for example the multivariate GARCH-type BEKK(1,1) model (Engle and Kroner, 1995) or the multivariate SNP model (Gallant and Tauchen, 1989): for our real multivariate example of section 4.3.2 with dimension $d = 7$, both models involve already more than 100 parameters which are likely to overfit (and searching for a best among all sparse parameterizations is combinatorially not manageable).

We have demonstrated on some data-sets (synthetic and real) that our FGD algorithm significantly outperforms the predictions from the CCC-GARCH(1,1) model (Bollerslev, 1990). The latter model has generated the starting functions in our FGD and hence, it is not so surprising that we could observe improvements. This seems generally the attractive feature of FGD. We choose a reasonable model for generating (estimating) the starting functions in FGD, and then we try – often successfully – to improve the initial basis model with a couple FGD iterations. According to the heuristics of a steepest functional gradient, the improvements with FGD are mainly expected at those components where the initial basis model performs poorly, as we demonstrated on a 100-dimensional data-set.

Our method is constructed from a generic algorithm: hence, other FGD procedures can be derived for learning in other, typically very high-dimensional, problems. Thus, FGD is not necessarily restricted to the framework of constant conditional correlation (CCC).

Appendix

When using regression tree base learners in FGD, we can take advantage of the fact that the learner is a partitioning method which allows a more (statistically) efficient implementation for fitting with respect to the best improvement of the loss function λ.

FGD with tree learners

(Formulated with response variables Y_i and explanatory variables X_i, see section 3).

Modified steps 2 and 3. Given a negative gradient vector U, fit a regression tree to U by least squares. This produces a partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_k\}$ of the predictor space $\mathbb{R}^p$

$$\cup_{j=1}^{k} \mathcal{R}_j = \mathbb{R}^p, \ \mathcal{R}_i \cap \mathcal{R}_j = \emptyset \ (i \neq j).$$

(I.e., the partition is such that $\sum_{i=1}^{n}(U_i - \sum_{j=1}^{k} \hat{\beta}_j \mathbb{I}_{[X_i \in \mathcal{R}_j]})^2$ is minimal). Then, proceed with line searches for all k partition cells,

$$\hat{\gamma}_{m,j} = \mathrm{argmin}_\gamma \sum_{i; X_i \in \mathcal{R}_j} \lambda(Y_i, \hat{F}_{m-1}(X_i) + \gamma \mathbb{I}_{[X_i \in \mathcal{R}_j]}), \ j = 1, \ldots, k.$$

Finally, up-date

$$\hat{F}_m(x) = \hat{F}_{m-1}(x) + \sum_{j=1}^{k} \hat{\gamma}_{m,j} \mathbb{I}_{[x \in \mathcal{R}_j]}.$$

This is the same generic tree algorithm as in Friedman (2001).

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