Applied Time Series Analysis
FS 2011 – Week 12

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Spectral Analysis

Idea: Time series are interpreted as a combination of cyclic components, and thus, a linear combination of harmonic oscillations.

Why: As a descriptive means, showing the character and the dependency structure within the series.

What: It is in spirit, but also mathematically, closely related to the correlogram

Where:
- engineering
- economics
- biology/medicine

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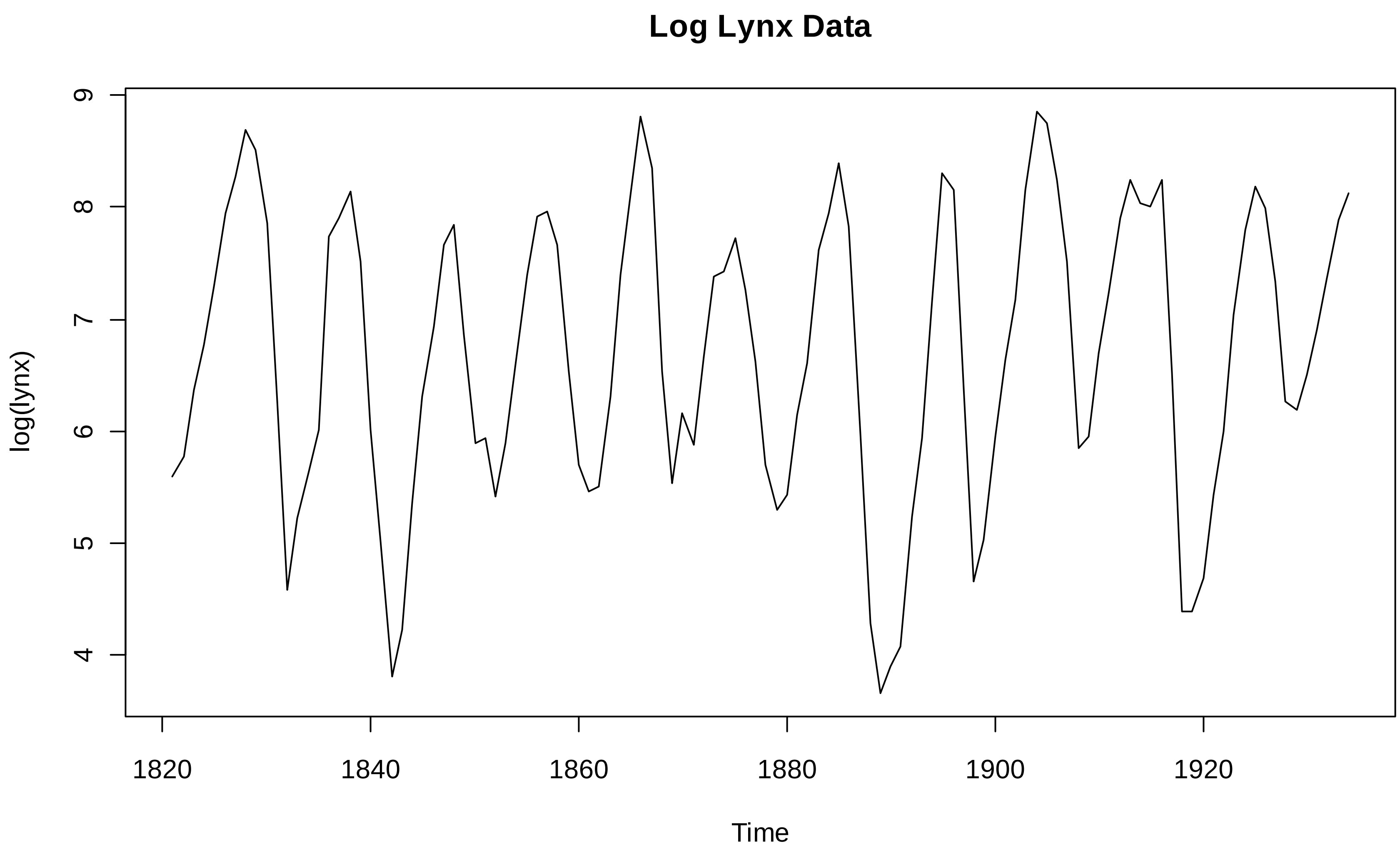
Lynx Data



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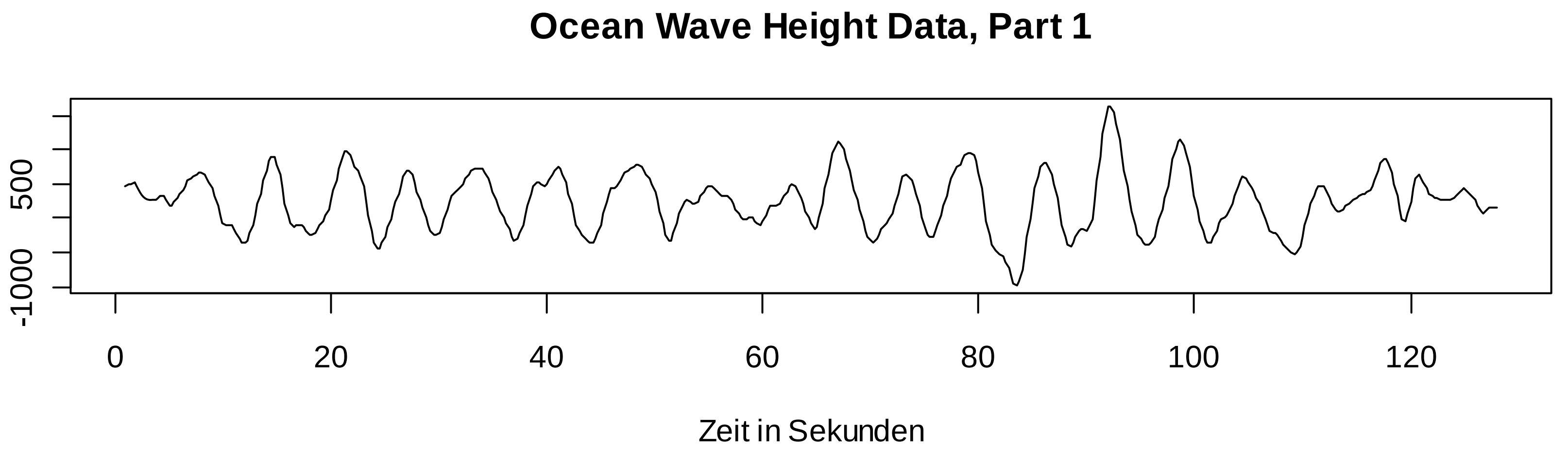
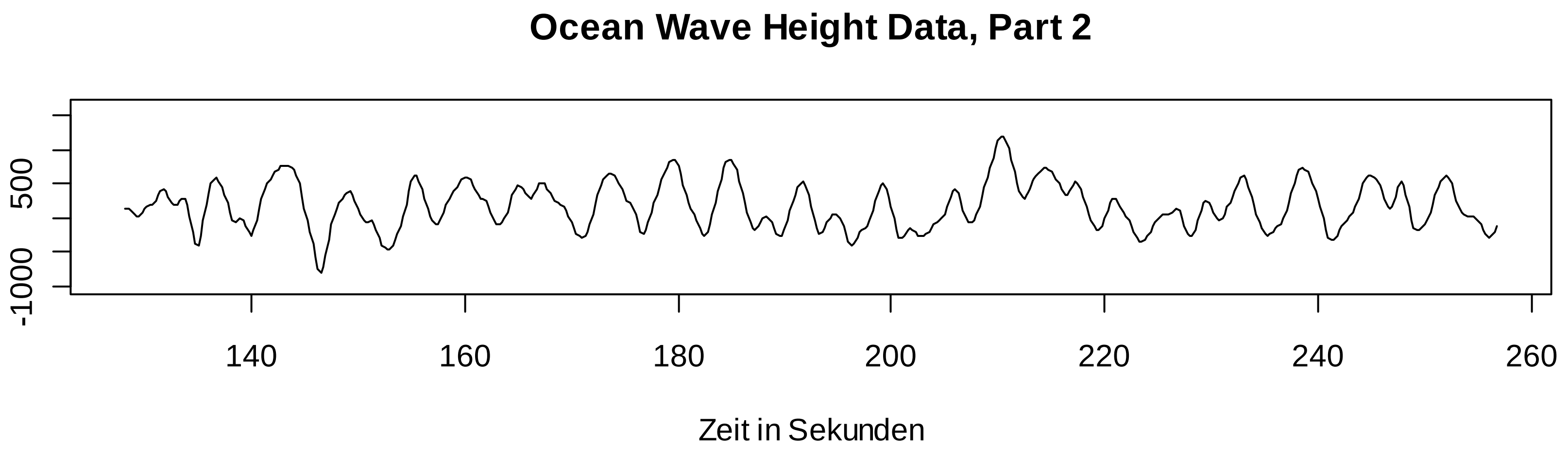
Ocean Wave Data





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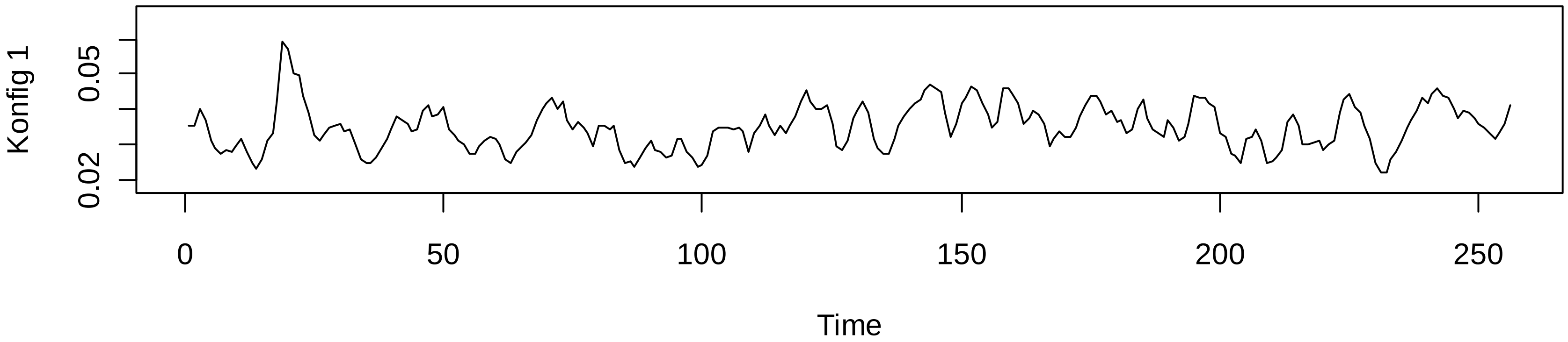
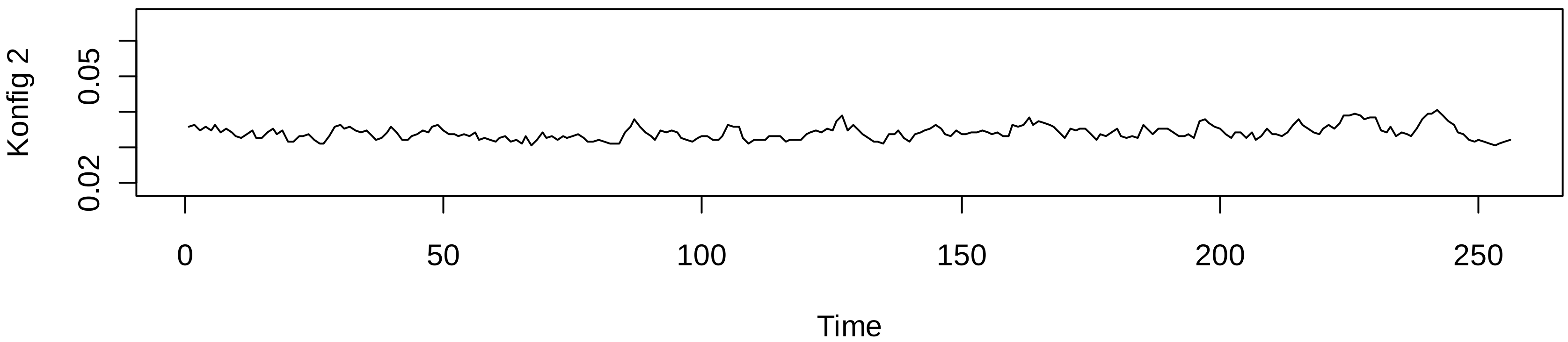
2-Component-Mixture Data

2-Component-Mixture: Series 1



2-Component-Mixture: Series 2



Harmonic Oscillations

The most simple periodic functions are sine and cosine, which we will use as the basis of our analysis.

A harmonic oscillation has the following form:

$$y(t) = \alpha \cos(2\pi\nu t) + \beta \sin(2\pi\nu t)$$

For the derivation, **see the blackboard…**

- In discrete time, we have aliasing, i.e. some frequencies cannot be distinguished (→ see next slide).
- The periodic analysis is limited to frequencies between 0 and 0.5, i.e. things we observe at least twice.

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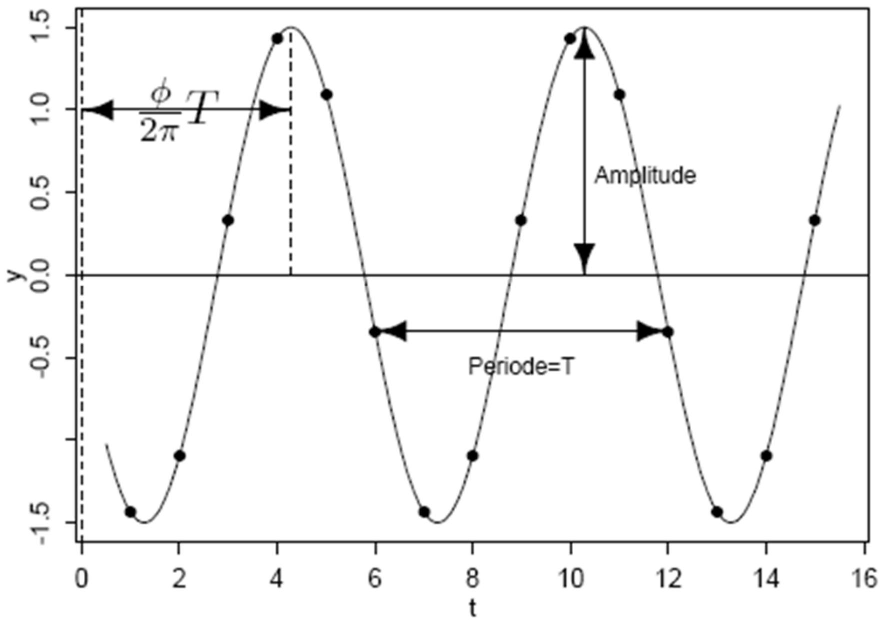
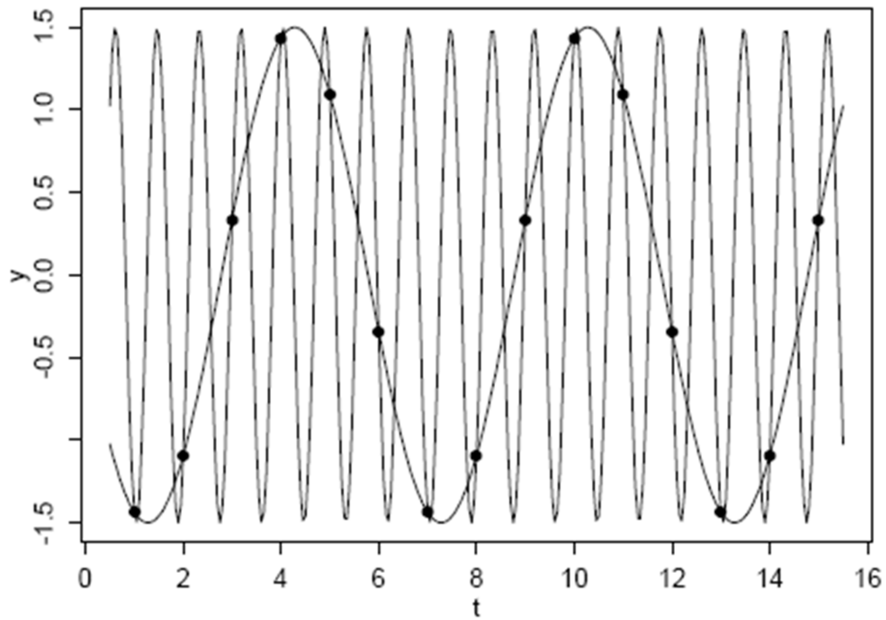
Aliasing





Regression Model & Periodogram

We try to write a time series with a regression equation containing sine and cosine terms at the fourier frequencies.

→ **see the blackboard**

The most important frequencies within the series, which when omitted, lead to pronounced increase in goodness-of-fit.

- This idea is used as a proxy for the periodogram,
  → **see the blackboard…**

- However, if the „true" frequency is not a fourier frequency, we have leakage (→ see next 2 slides).

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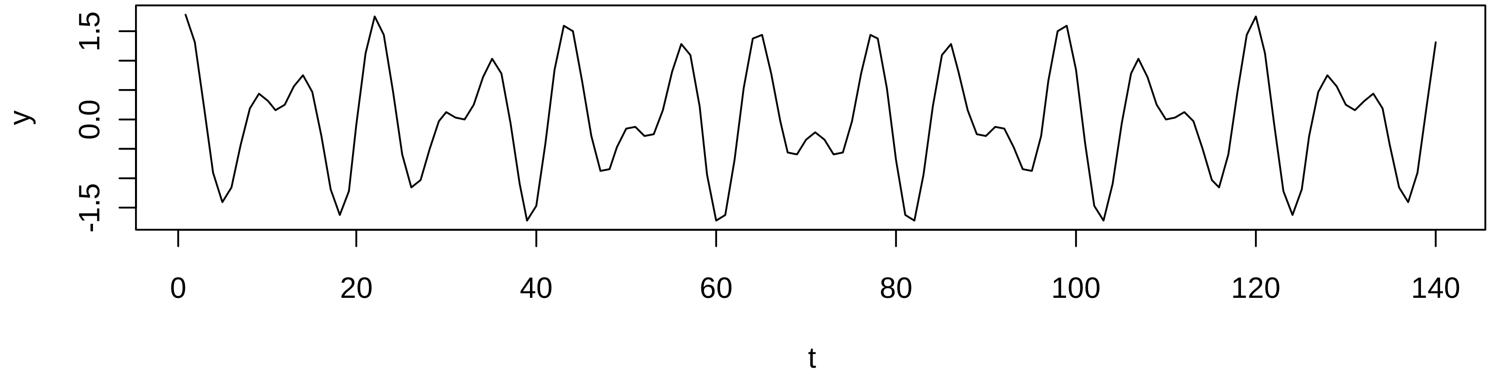
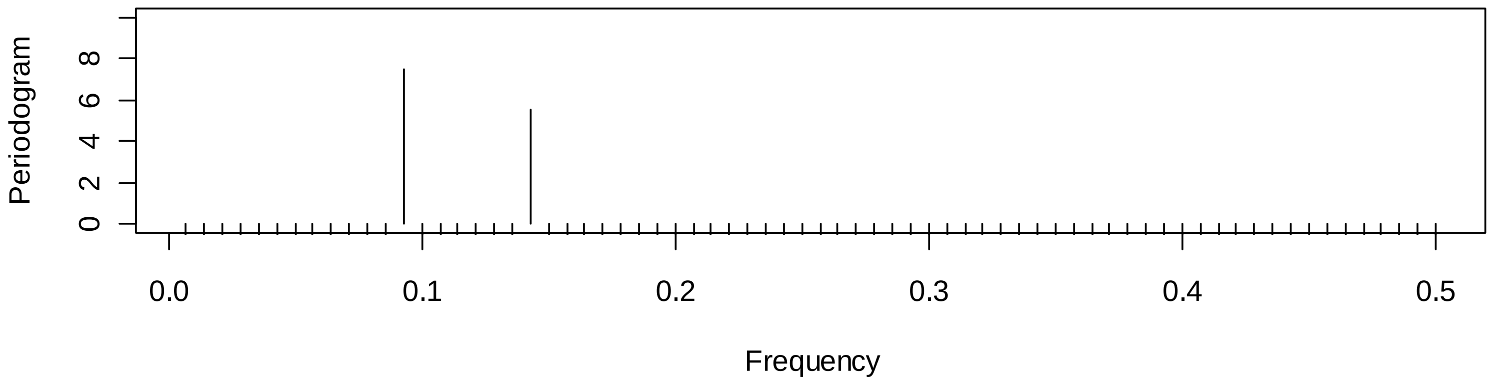
Periodogram of a Simulated Series

Simulated Series



Periodogram of the Series



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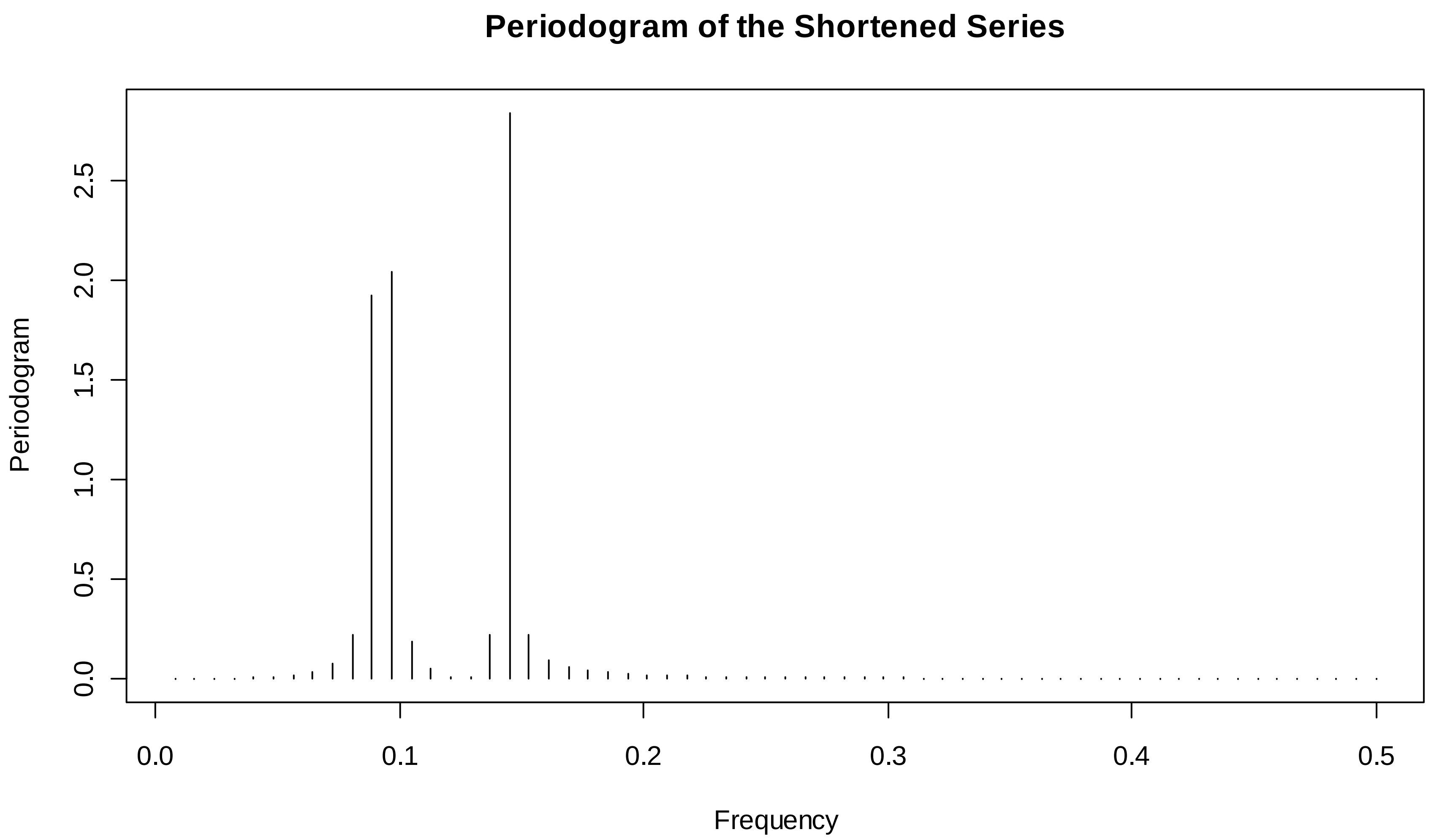
Periodogram of the Shortened Series



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Properties of the Periodogram

Periodogram and correlogram are mathematically equivalent, the former is the **fourier transform** of the latter.

→ **see the blackboard** for the derivation

Note: this is a reason why we divided by 1/n in the ACV.

- $I(\nu_k)$ or $\log(I(\nu_k))$ are plotted against k/n
- Estimates seem rather instable and noisy
- On the log-scale, most frequencies are present
- It seems as if smoothing is required for interpretation.

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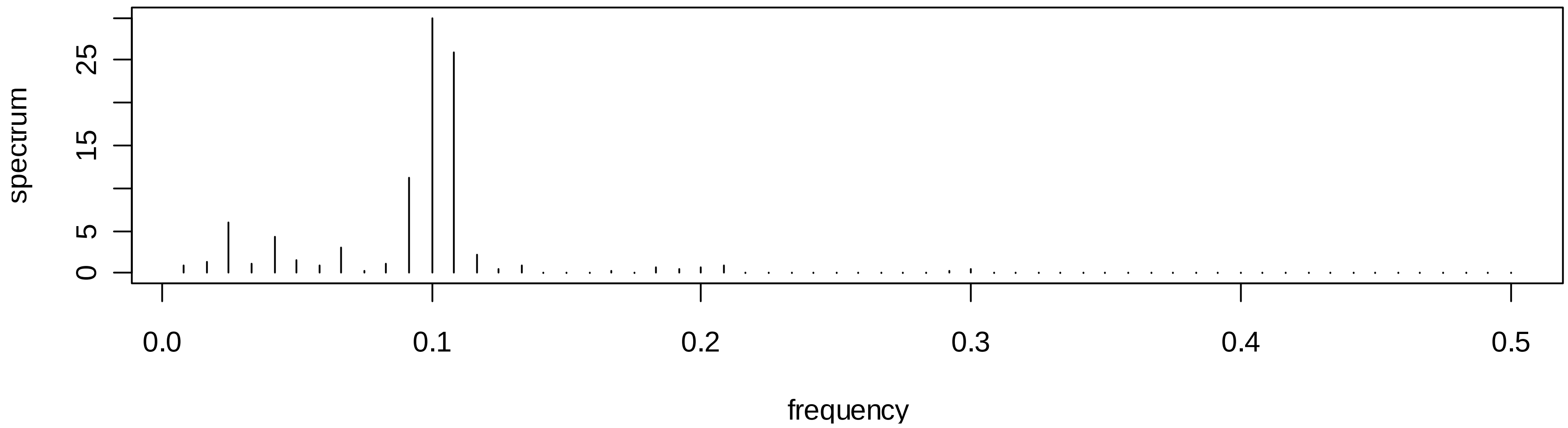
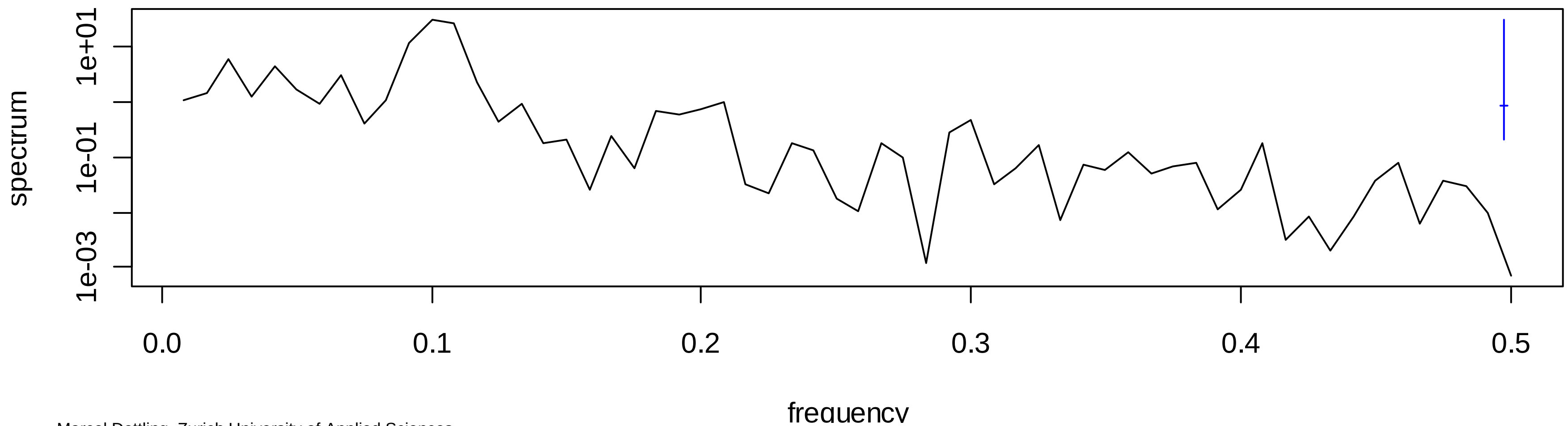
Periodogram of the Log Lynx Data





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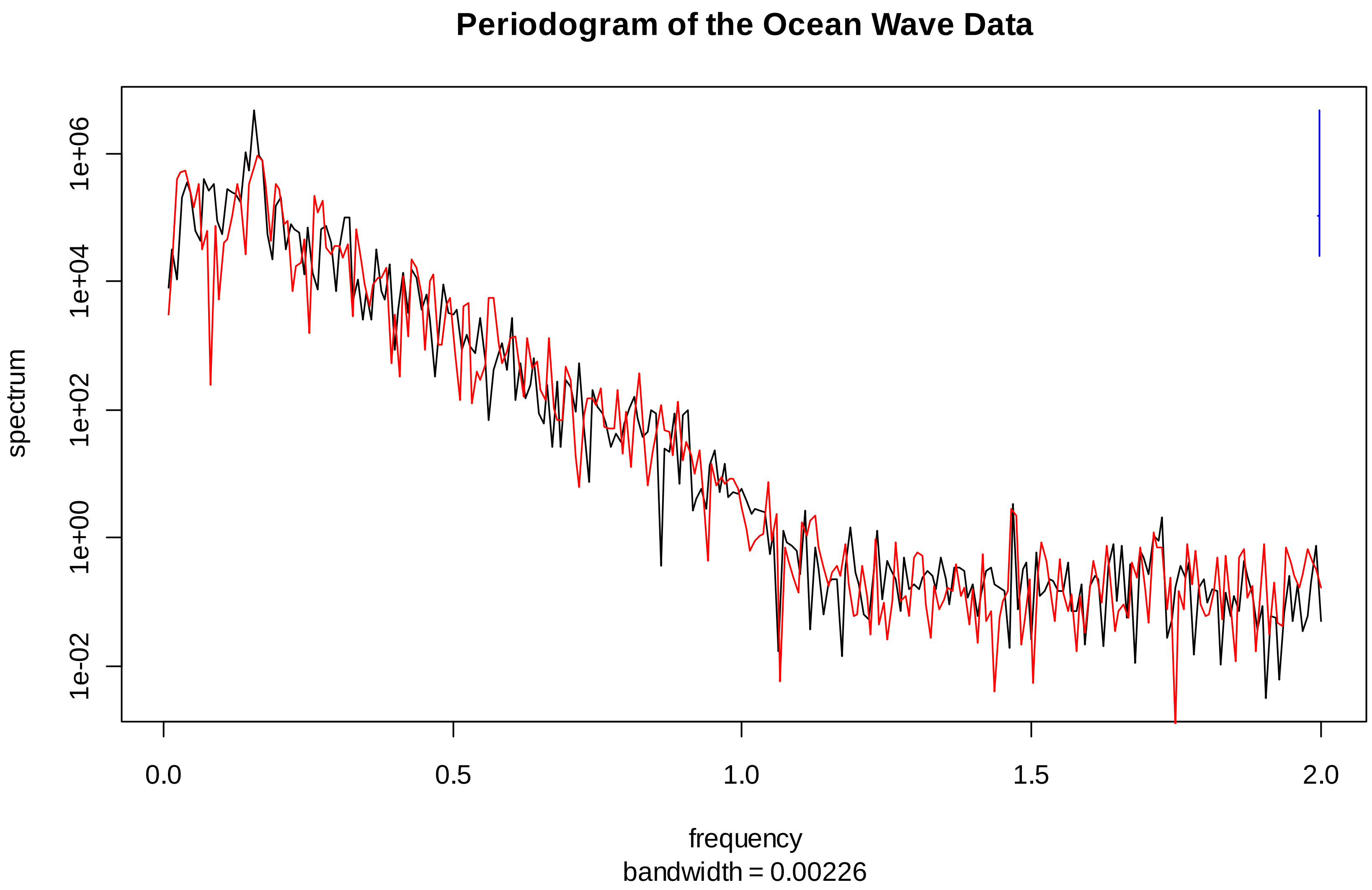
Periodogram of the Ocean Wave Data



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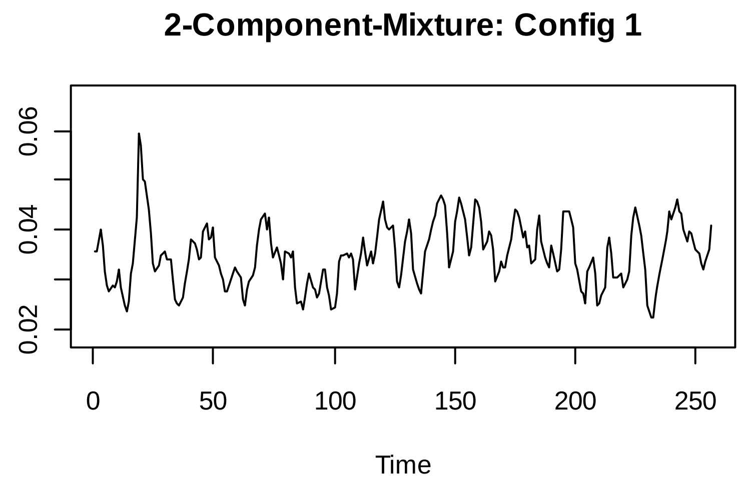
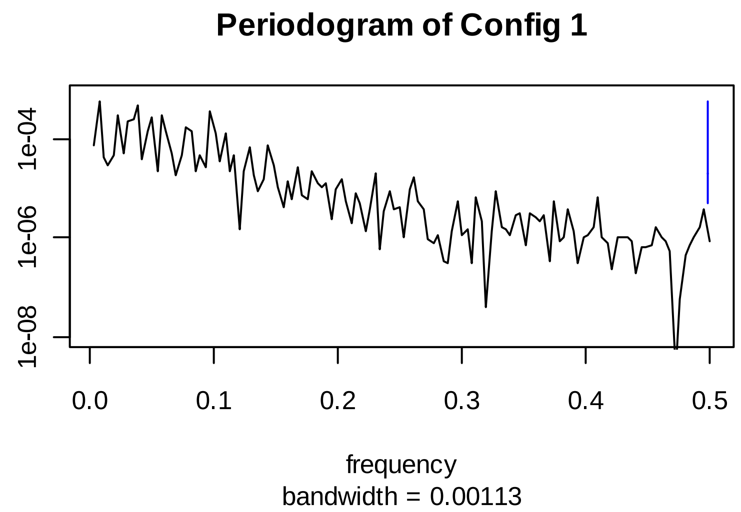
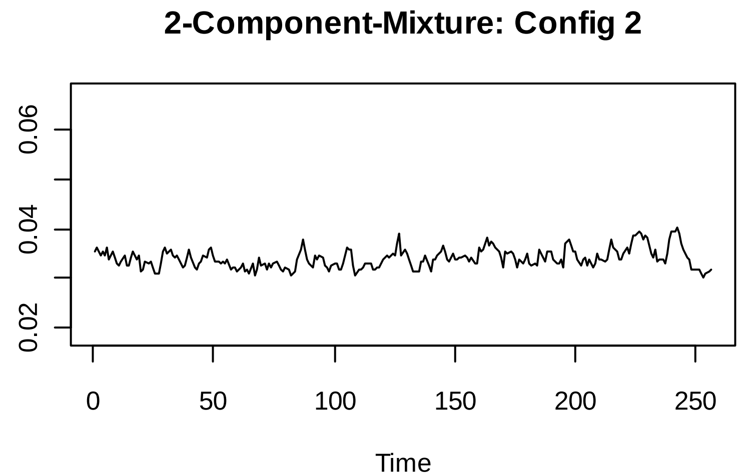
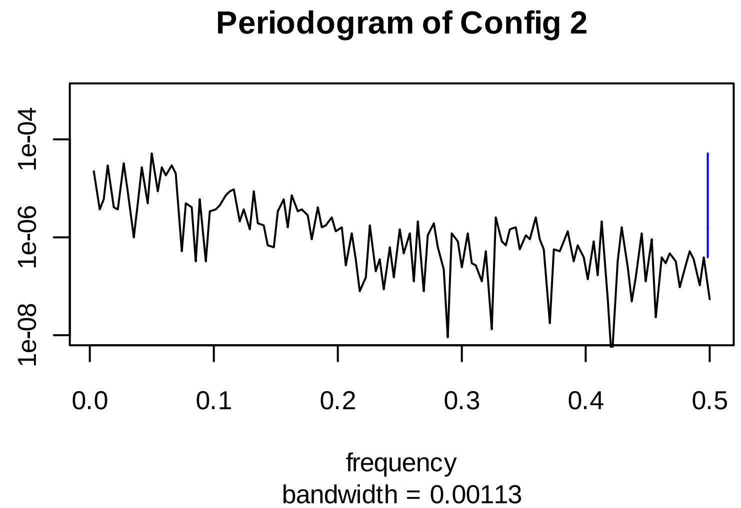
Periodogram of the 2-Component-Mixture









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The Spectrum

Observed time series	⇔	Stochastic process
Empirical ACF	⇔	Theoretical ACF
Periodogram	⇔	Spectrum

There is a link between ACF and periodogram/spectrum

$$f(\nu) = \sum_{k=-\infty}^{+\infty} \gamma(k)\cos(2\pi\nu k)$$

and

$$\gamma(k) = \int_{-0.5}^{+0.5} f(\nu)\cos(2\pi\nu k)d\nu$$

respectively. The spectrum is thus the Fourier transformation of the ACV.

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What‘s the Spectrum Good For?

Theorem: **Cramer Representation**

Every stationary process can be written as the limit of a linear combination consisting of harmonic oscillations with random, uncorrelated amplitudes.

- The spectrum characterizes the variance of all these random amplitudes.
- Or vice versa: $\int_{\nu_1}^{\nu_2} f(\nu)d\nu$ is the variance between the frequencies that make the integration limits.
- The spectrum takes only positive values. Thus, not every ACF sequence defines a stationary series.

A Few Particular Spectra

- **White noise**
  → the spectrum is constant over all frequencies.

- **AR(1),** see next slide
  → already quite a complicated function α_1

- **ARMA (p,q)**
  → the characteristic polynoms determine the spectrum

$$f(\nu) = \sigma_E^2 \frac{|\Theta(\exp(-i2\pi\nu))|}{|\Phi(\exp(-i2\pi\nu))|}$$

- Note: to generate m maxima in the spectrum, we require an AR-model, where the order is at least $2m$.

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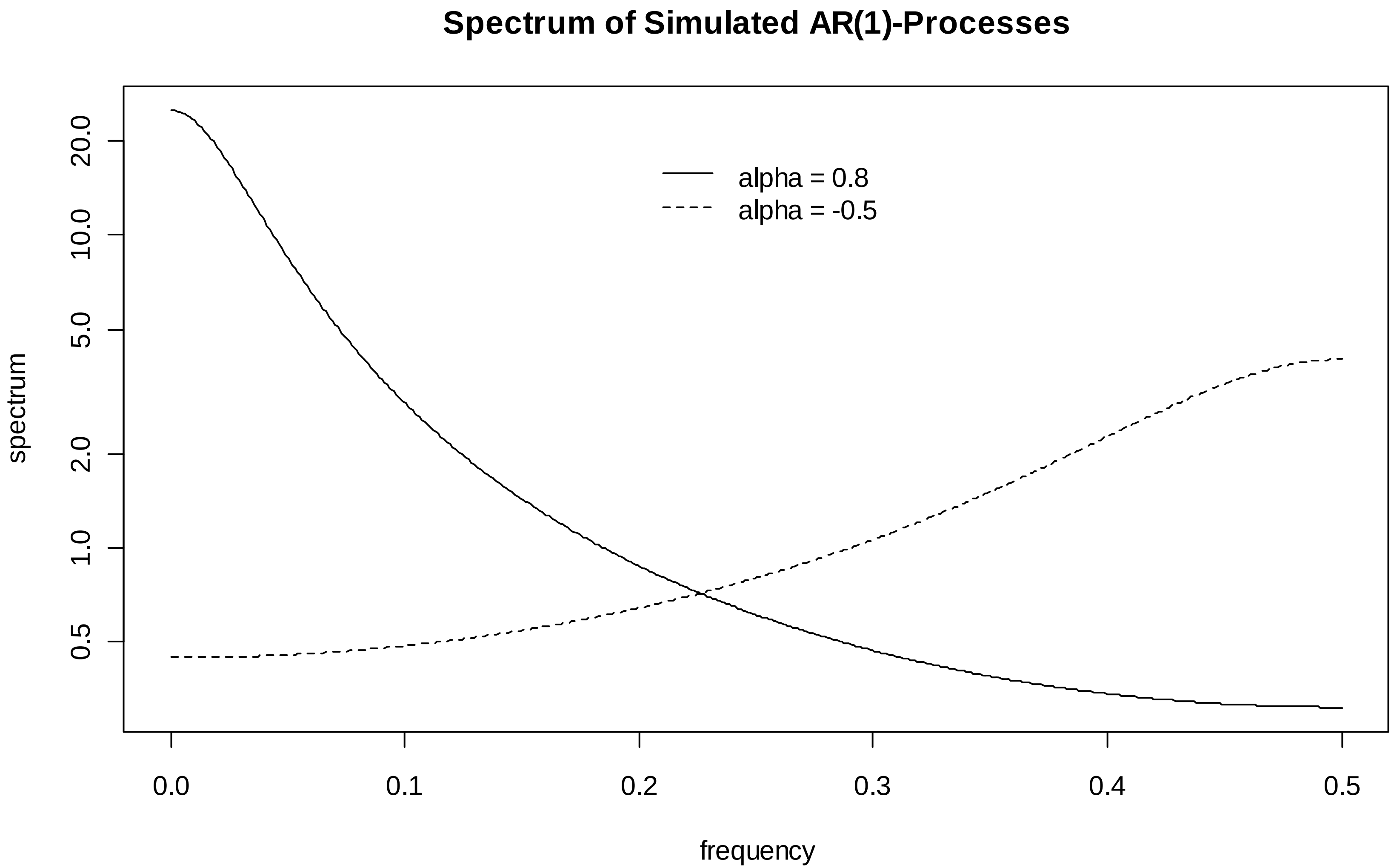
Spectrum of AR(1)-Processes



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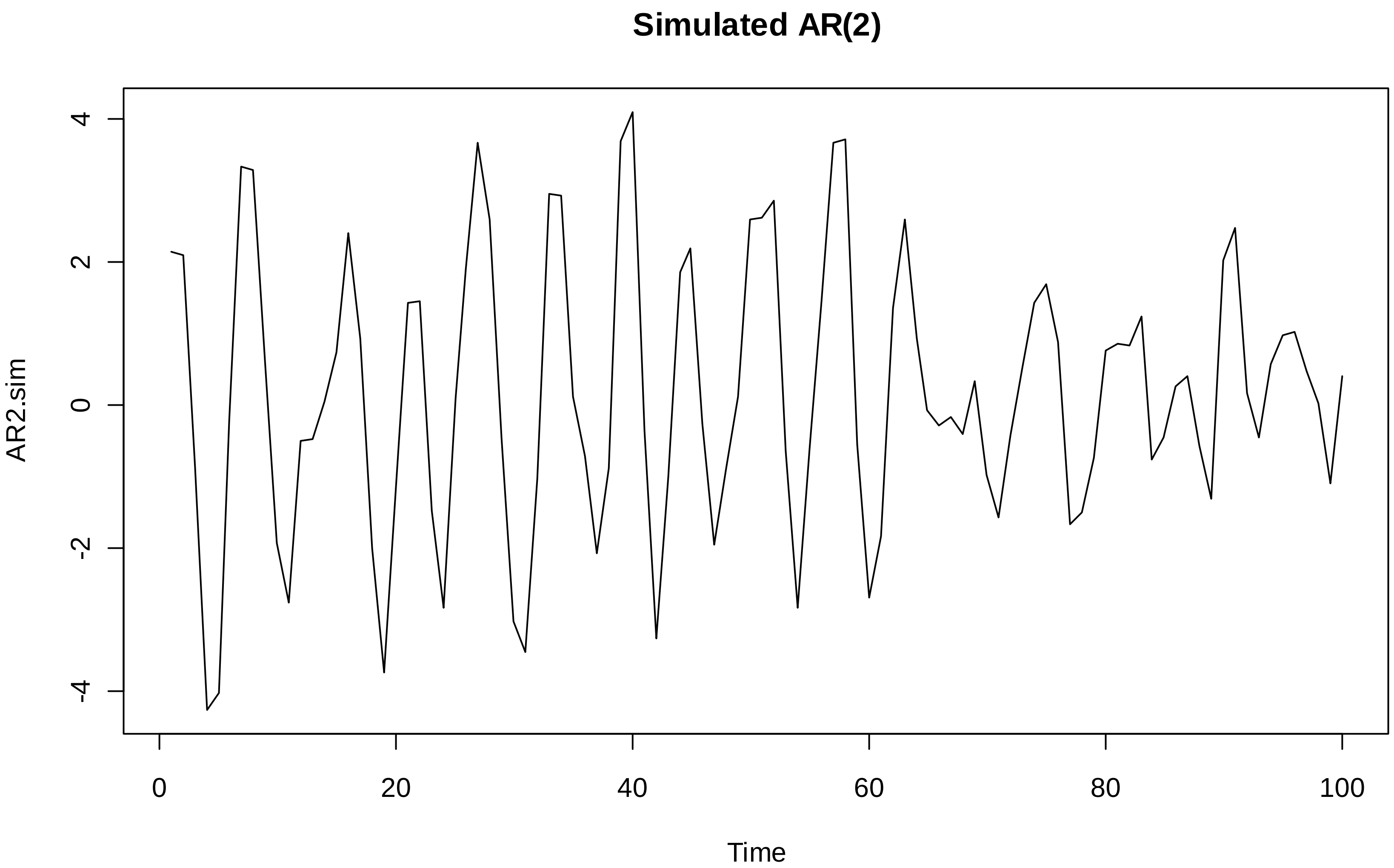
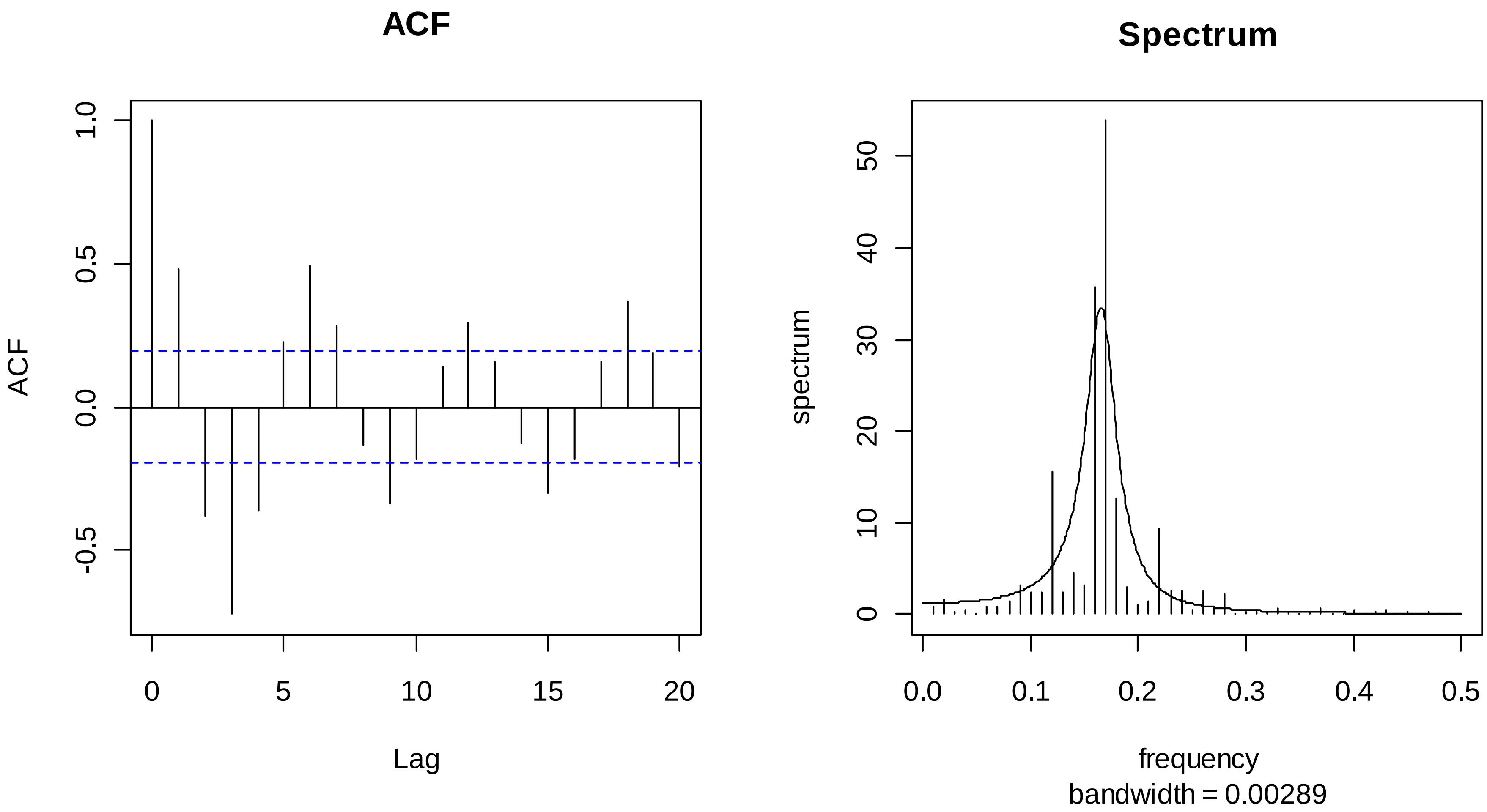
Simulated AR(2)-Process



ACF/Spectrum of Simulated AR(2)-Process



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Spectral Analysis

- **Spectral analysis** is a descriptive technique, where the time series is interpreted as a linear combination of harmonic oscillations.
- The **periodogram** shows empirically, which frequencies are „important“, i.e. lead to a substantial increase in RSS when ommitted from the linear combination.
- The **spectrum** is the theoretical counterpart to the periodogram. It can also be seen as the Fourier transformation of the theoretical autocovariances.
- The periodogram is a poor **estimator** for the spectrum: it‘s not smooth and inconsistent.

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Improving the Raw Periodogram

1) Smoothing with a running mean

+ simple approach
- choice of the bandwith

2) Smoothing with a weighted running mean

+ choice of the bandwith is less critical
- difficulties shift to the choice of weights

3) Weighted plug-in estimation

+ weighted Fourier trsf. of estimated autocovariances
- choice of weights

4) Piecewise periodogram estimation with averaging

+ can serve as a check for stationarity, too

Improving the Raw Periodogram

5) Spectrum of an estimated model

+ fundamentally different from 1)-4)
- only works for „small“ orders p

6) Tapering

+ further modification of periodogram estimation
+ reduces the bias in the periodogram
+ should always be applied

7) Prewhitening and Rescoloring

+ model fit and periodogram estimation on residuals
+ the effect of the model will be added again

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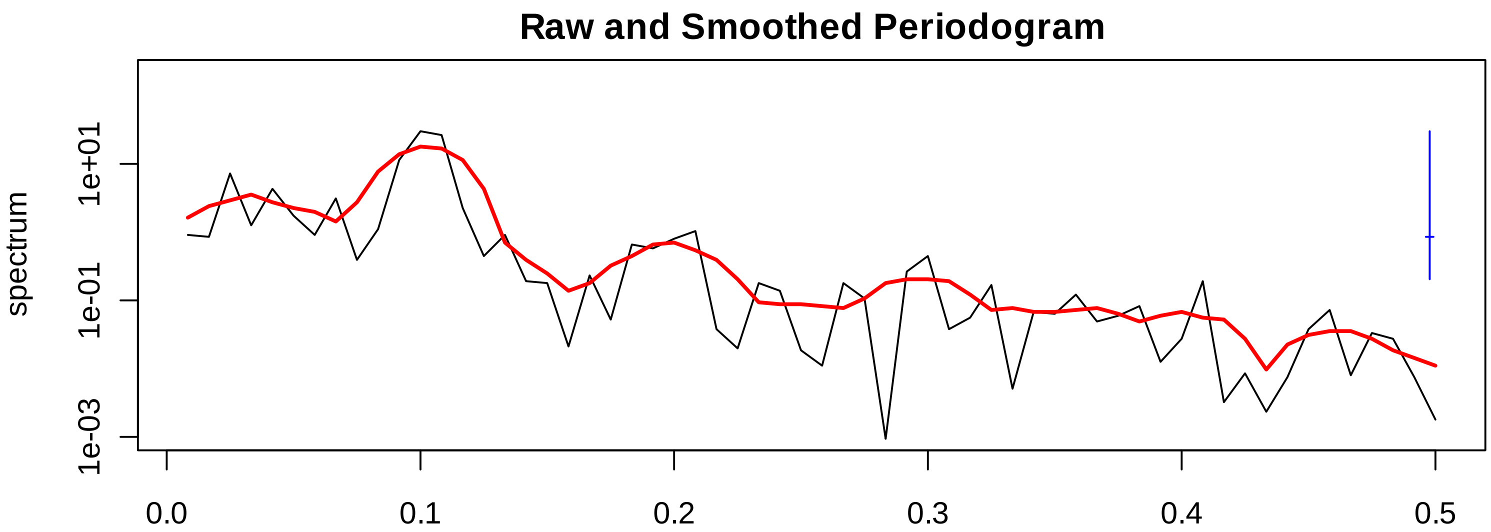
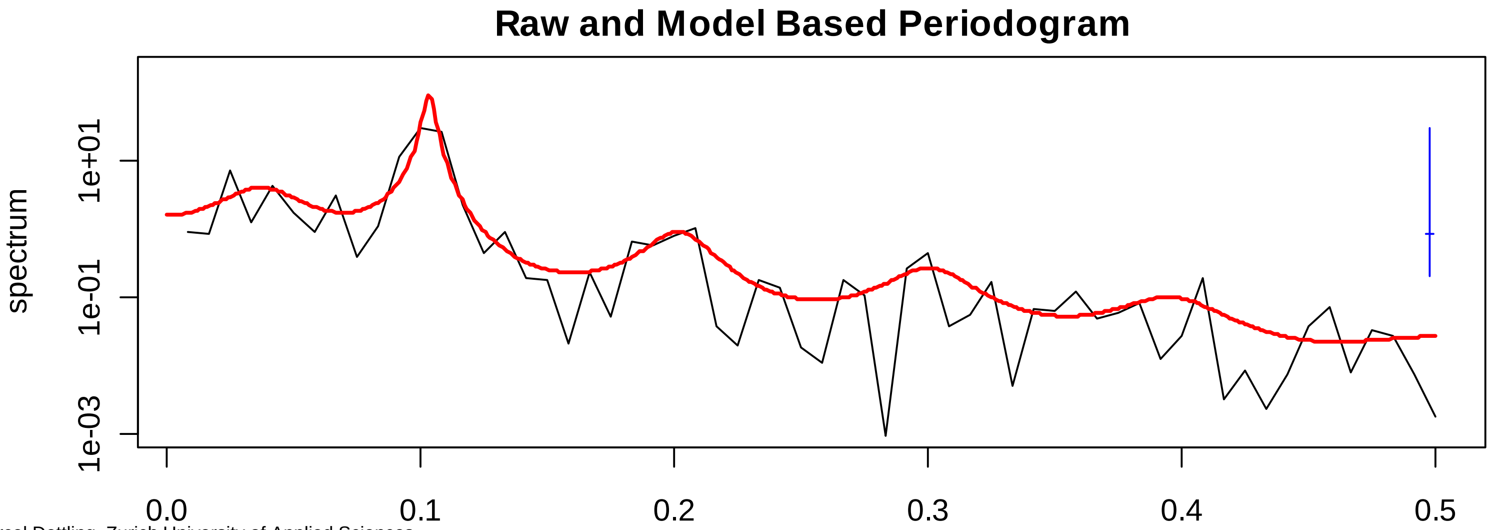
Modified Periodogram of log(Lynx) Data





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