Applied Time Series Analysis
FS 2011 – Week 11

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Prewhitening the Gas Furnace Data

What to do:

- AR(p)-models are fitted to the differenced series
- The residual time series are U_t and V_t, white noise
- Check the sample cross correlation (see next slide)
- Model the output as a linear combination of past input values: transfer function model.

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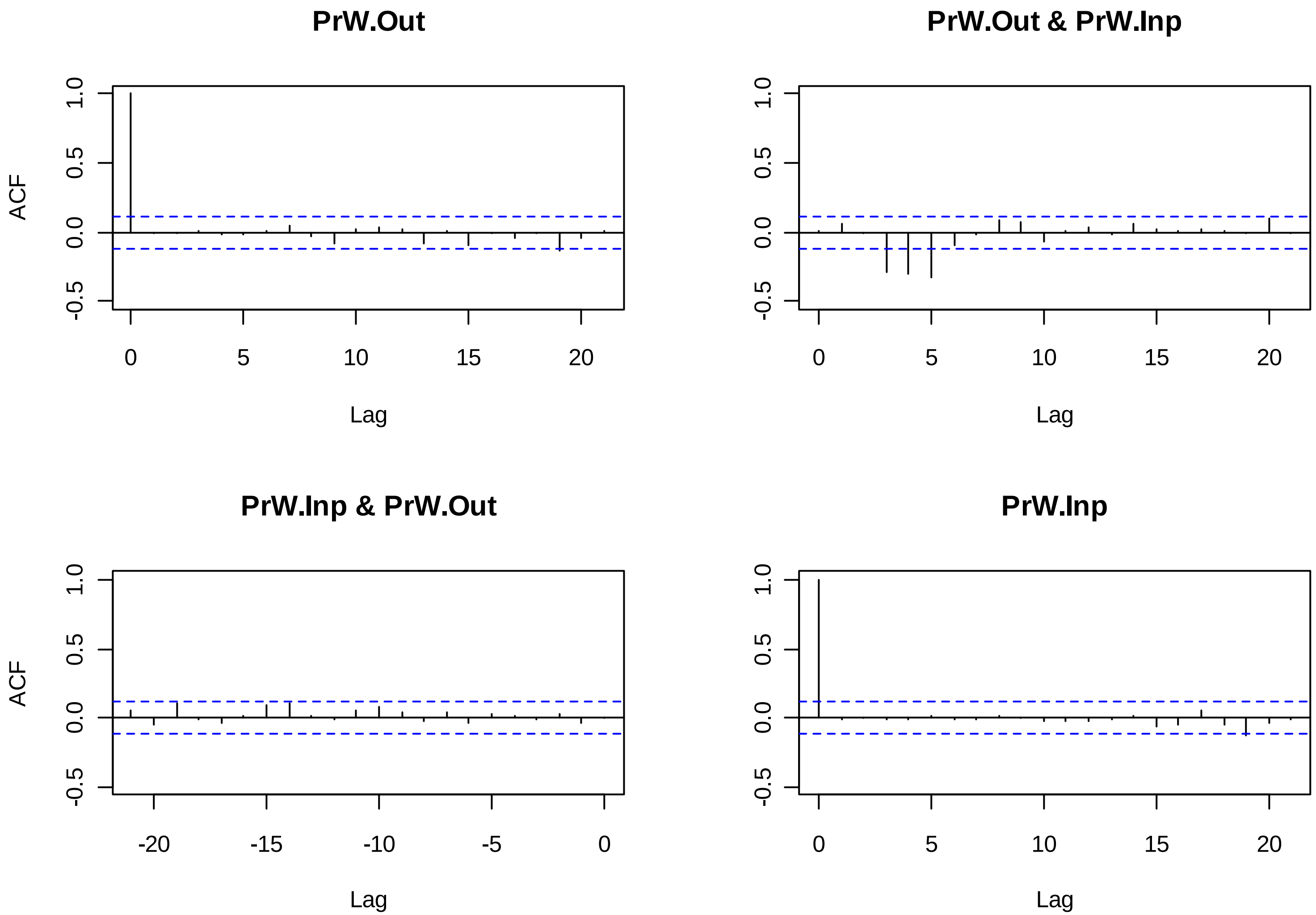
Prewhitening the Gas Furnace Data



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Transfer Function Models

Properties:

- Transfer function models are an option to describe the dependency between two time series.
- The first (input) series influences the second (output) one, but there is no feedback from output to input.
- The influence from input to output only goes „forward“.

The model is:

$$X_{2,t} - \mu_2 = \sum_{j=0}^{\infty} \nu_j (X_{1,t-j} - \mu_1) + E_t$$

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Transfer Function Models

The model is:

$$X_{2,t} - \mu_2 = \sum_{j=0}^{\infty} \nu_j (X_{1,t-j} - \mu_1) + E_t$$

- $E[E_t]=0$.
- E_t and $X_{1,s}$ are uncorrelated for all t and s.
- E_t and E_s are usually correlated.
- For simplicity of notation, we here assume that the series have been mean centered.

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Cross Covariance

When plugging-in, we obtain for the cross covariance:

$$\gamma_{21}(k) = Cov(X_{2,t+k}, X_{1,t}) = Cov\left(\sum_{j=0}^{\infty} \nu_j X_{1,t+k-j}, X_{1,t}\right) = \sum_{j=0}^{\infty} \nu_j \gamma_{11}(k-j)$$

- If only finitely many coefficients are different from zero, we could generate a linear equation system, plug-in $\hat{\gamma}_{11}$ and $\hat{\gamma}_{21}$ to obtain the estimates $\hat{\nu}_j$.

→ **This is not a statistically efficient estimation method.**

Special Case: $X_{1,t}$ Uncorrelated

If $X_{1,t}$ was an uncorrelated series, we would obtain the coefficients of the transfer function model quite easily:

$$\nu_k = \frac{\gamma_{21}(k)}{\gamma_{11}(0)}$$

However, this is usually not the case. We can then:

- transform all series in a clever way
- the transfer function model has identical coefficients
- the new, transformed input series is uncorrelated

→ **see blackboard for the transformation…**

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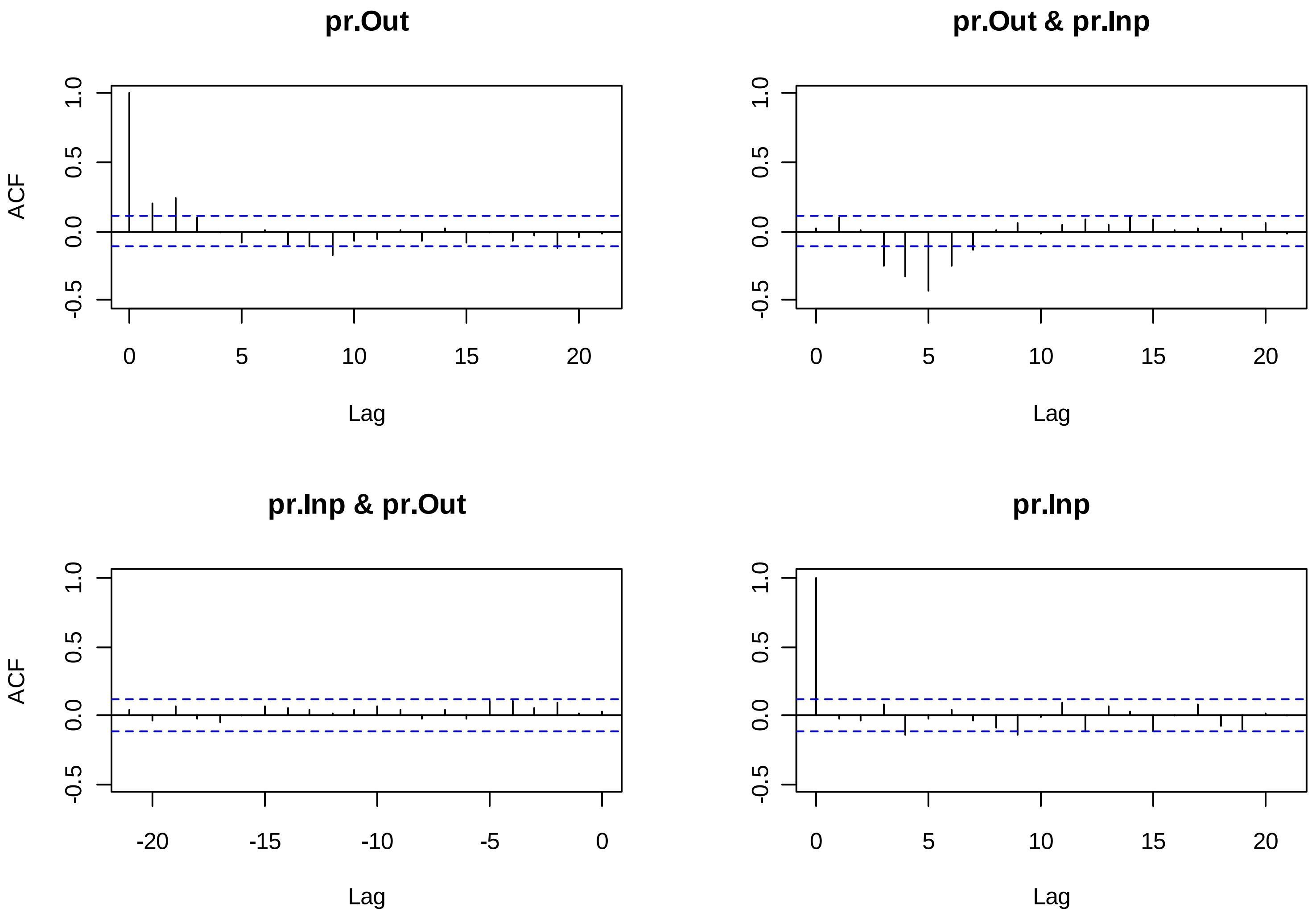
Gas Furnace Transformed



Gas Furnace: Final Remarks

- In the previous slide, we see the empirical cross correlations $\hat{\rho}_{21}(k)$ of the two series D_t and Z_t.
- The coefficients $\hat{v}_k$ from the transfer function model will be proportional to the empirical cross correlations. We can already now conjecture that the output is delayed by 3-7 times, i.e. 27-63 seconds.
- The formula for the transfer function model coefficients is:

$$\hat{v}_k = \frac{\hat{\sigma}_Z}{\hat{\sigma}_D} \hat{\rho}_{21}(k)$$

State Space Models

Basic idea: There is a stochastic process/time series Z_t which we cannot directly observe, but only under the addition of some white noise.

Thus: We observe the time series

$$Y_t = Z_t + U_t \quad \text{where} \quad U_t \sim N(0, \tau^2)\text{, i.i.d.}$$

Example: Z_t = # of fish in a lake
Y_t = # estimated number of fish from a sample

Notation and Terminology for an AR(1)

We assume that the true underlying process is an AR(1), i.e.

$$Z_t = a_1 Z_{t-1} + E_t \,,$$

where

$E_t \sim N(0, \sigma_E^2)$ are i.i.d. innovations, „process noise“.

In practice, we only observe y_t, as realizations of the process

$Y_t = Z_t + U_t$, with $U_t \sim N(0, \tau^2)$, i.i.d.

and additionally, the U_t are independent of Z_s, E_s for all s,t, thus they are independent „observation white noise“.

More Terminology

We call

$$Z_t = a_1 Z_{t-1} + E_t$$

the „state equation“, and

$$Y_t = Z_t + U_t$$

the „observation equation“.

On top of that, we remember once again that the „process noise“ E_t is an innovation that affects all future values Z_{t+k} and thus also Y_{t+k}, whereas U_t only influences the current observation Y_t, but no future ones.

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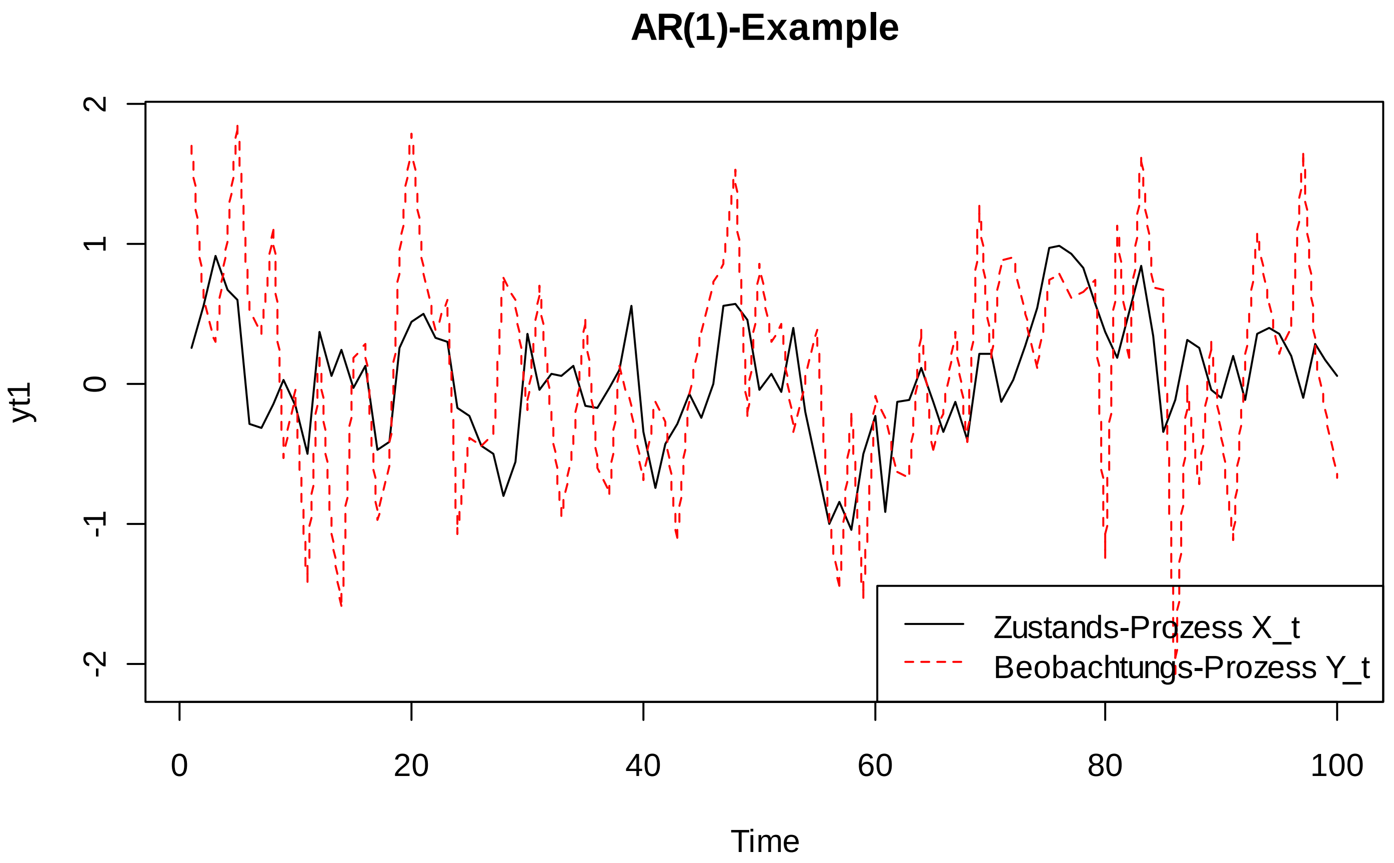
AR(1)-Example with α=0.7



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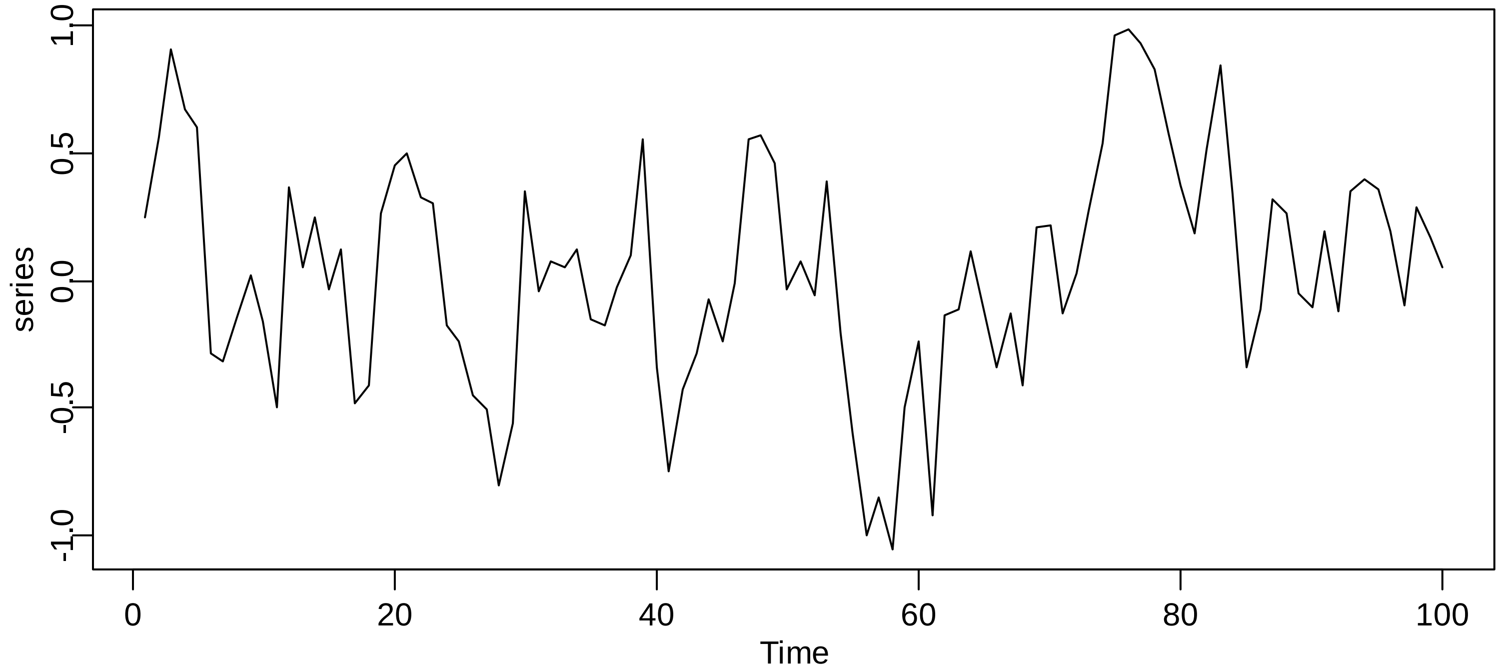
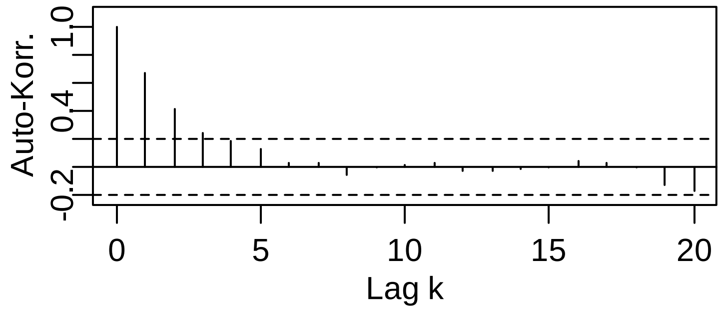
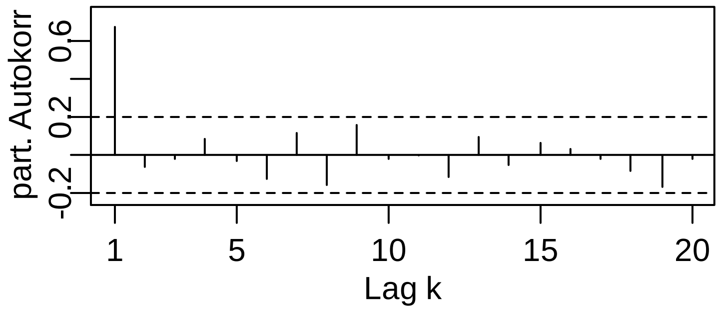
ACF/PACF of Z_t







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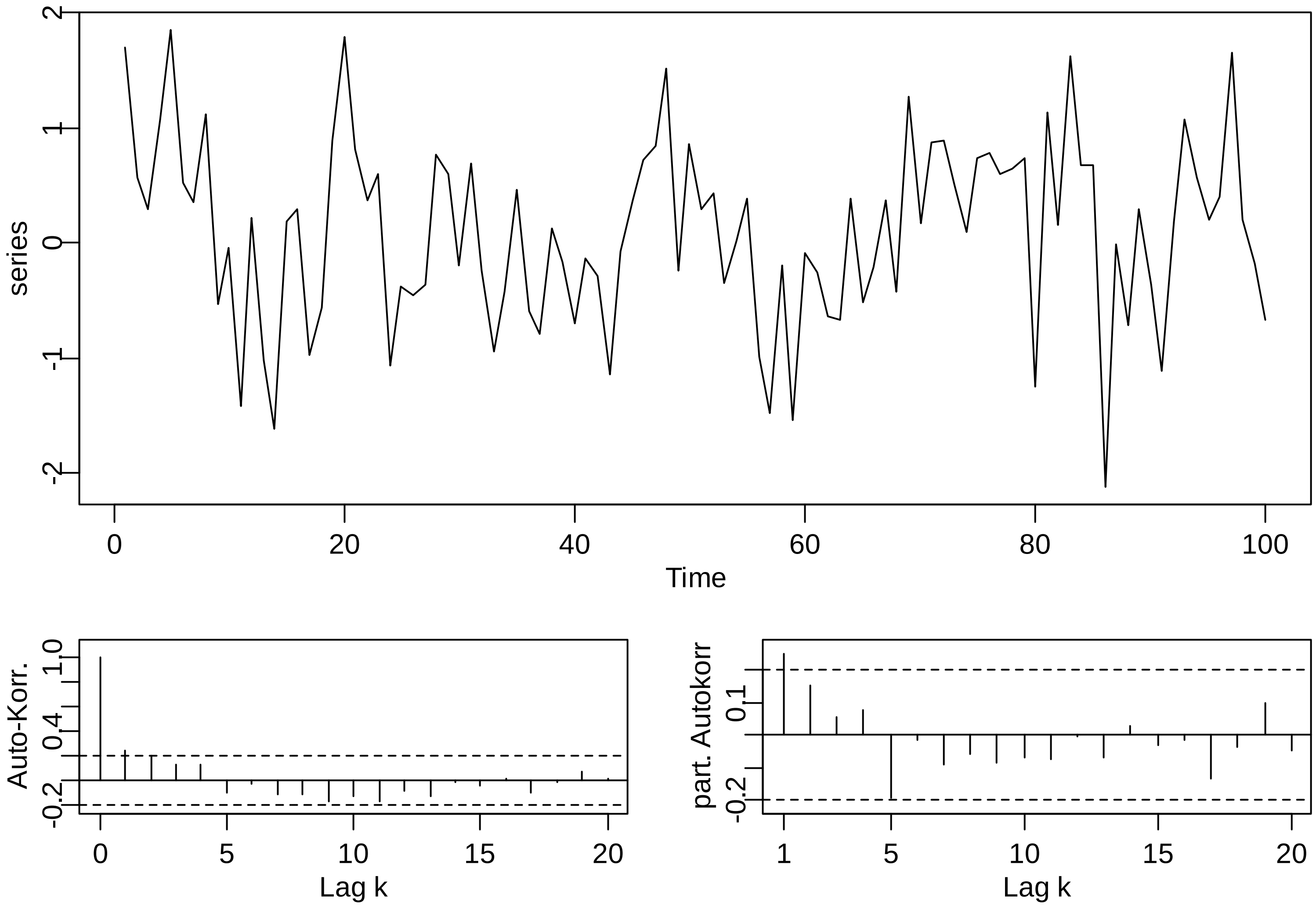
ACF/PACF of Y_t



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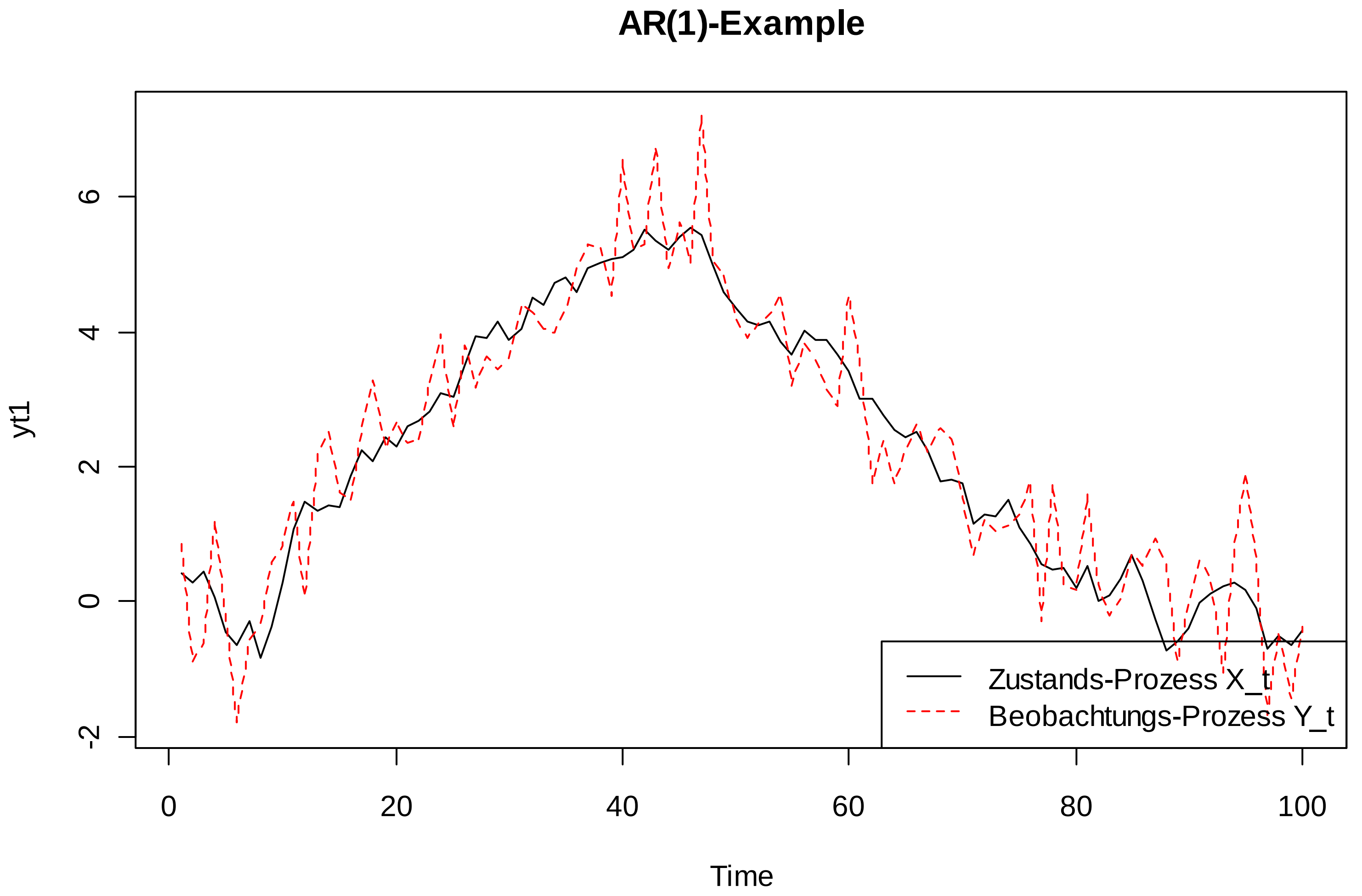
AR(1)-Example with α=1



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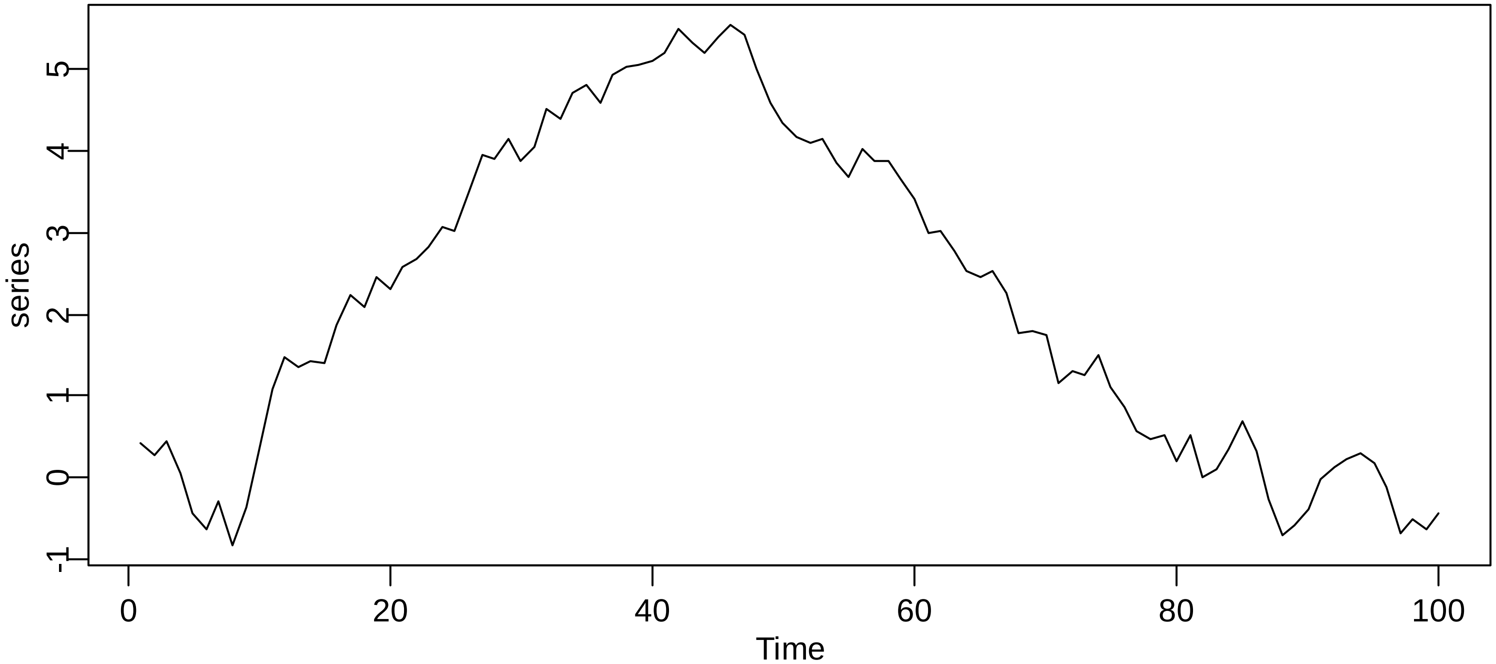
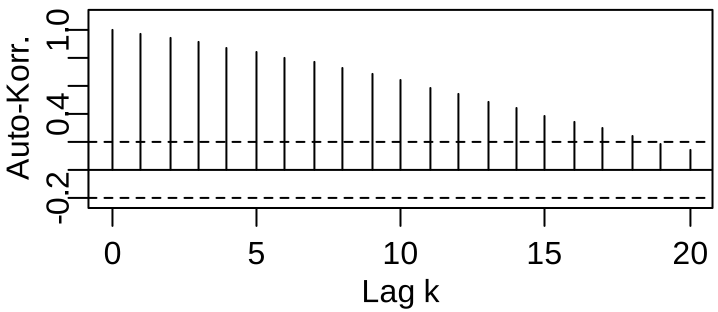
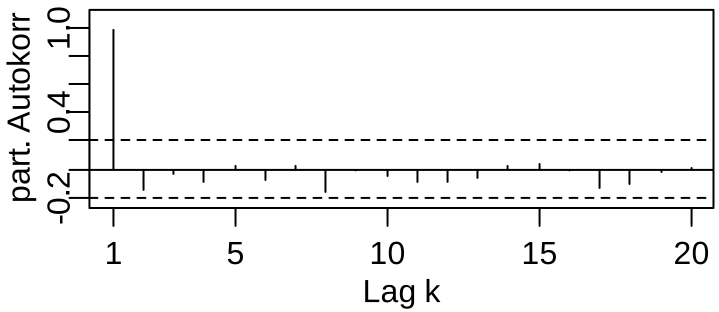
ACF/PACF of Z_t







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What is the goal?

The goal of State Space Modeling/Kalman Filtering is:

To uncover the „de-noised" process Z_t from the observed process Y_t.

- The algorithm of Kalman Filtering works with non-stationary time series, too.
- The algorithm is based on a maximum-likelihood-principle where one assume normal distortions.
- There are extensions to multi-dimensional state space models. This is partly discussed in the exercises.

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Summary of Kalman Filtering

Summary:

1) The Kalman Filter is a recursive algorithm

2) It relies on an update idea, i.e. we update the forecast $\hat{Z}_{t+1,t}$ with the difference $(y_{t+1} - \hat{Y}_{t+1,t})$.

3) The weight of the update is determined by the relation between the process variance σ_E^2 and the observation white noise τ^2.

4) This relies on the knowledge of g, h, σ_E^2, τ^2. In practice we have procedures for simultaneous estimation.

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Additional Remarks

1) For the recursive approach of Kalman filtering, initial values are necessary. Their choice is not crucial, their influence cancels out rapidly.

2) The procedures yield forecast and filter intervals:
$\hat{Z}_{t+1,t} \pm 1.96 \cdot \sqrt{R_{t+1,t}}$ and $\hat{Z}_{t+1,t+1} \pm 1.96 \cdot \sqrt{R_{t+1,t+1}}$

3) State space models are a very rich class. Every ARIMA(p,d,q) can be written in state space form, and the Kalman filter can be used for estimating the coefficients.

4) We can also use Kalman filtering for smoothing, i.e. providing $Z_t \mid Y_1^T$ with T>t.

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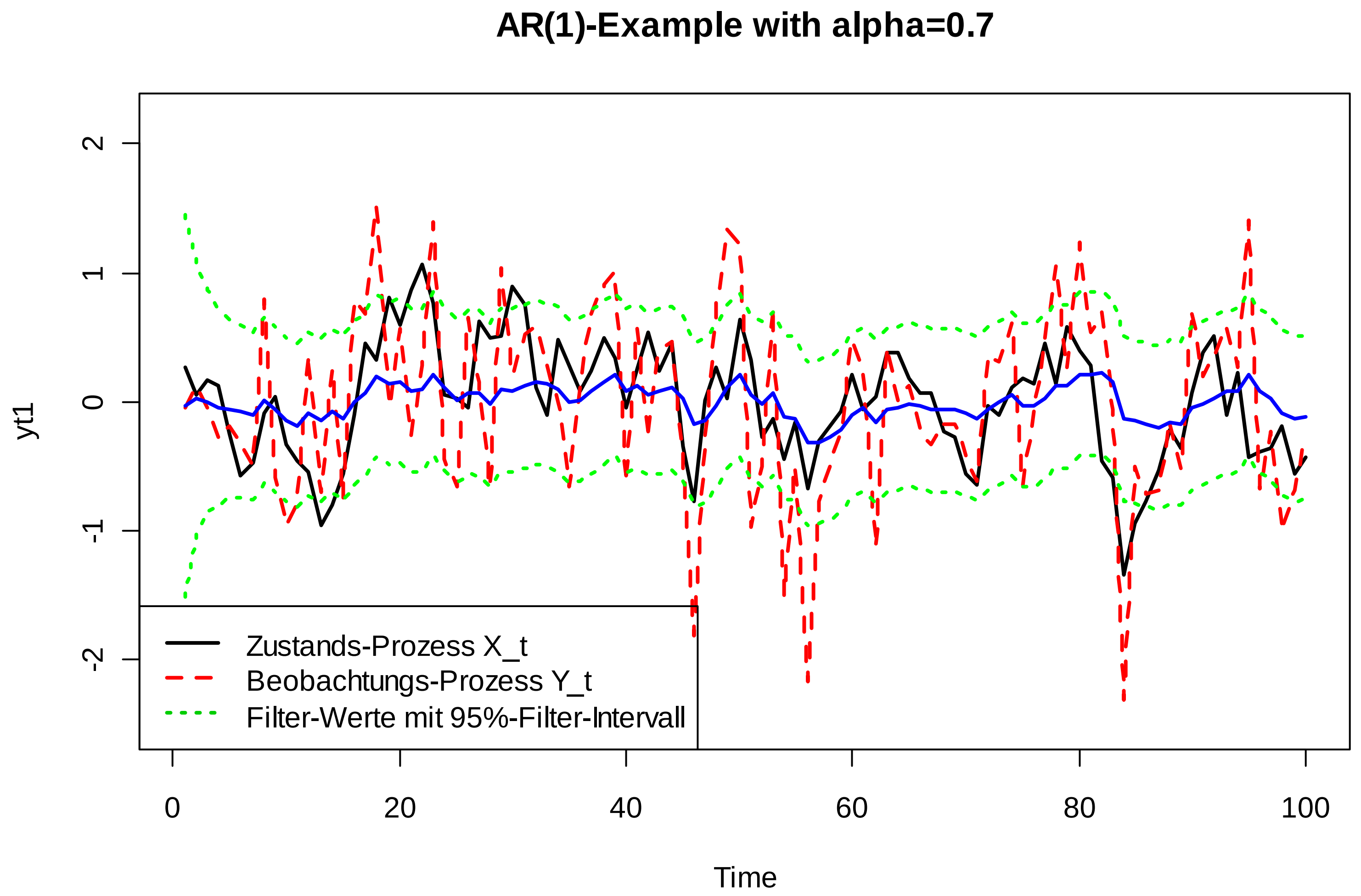
AR(1)-Example with α=0.7



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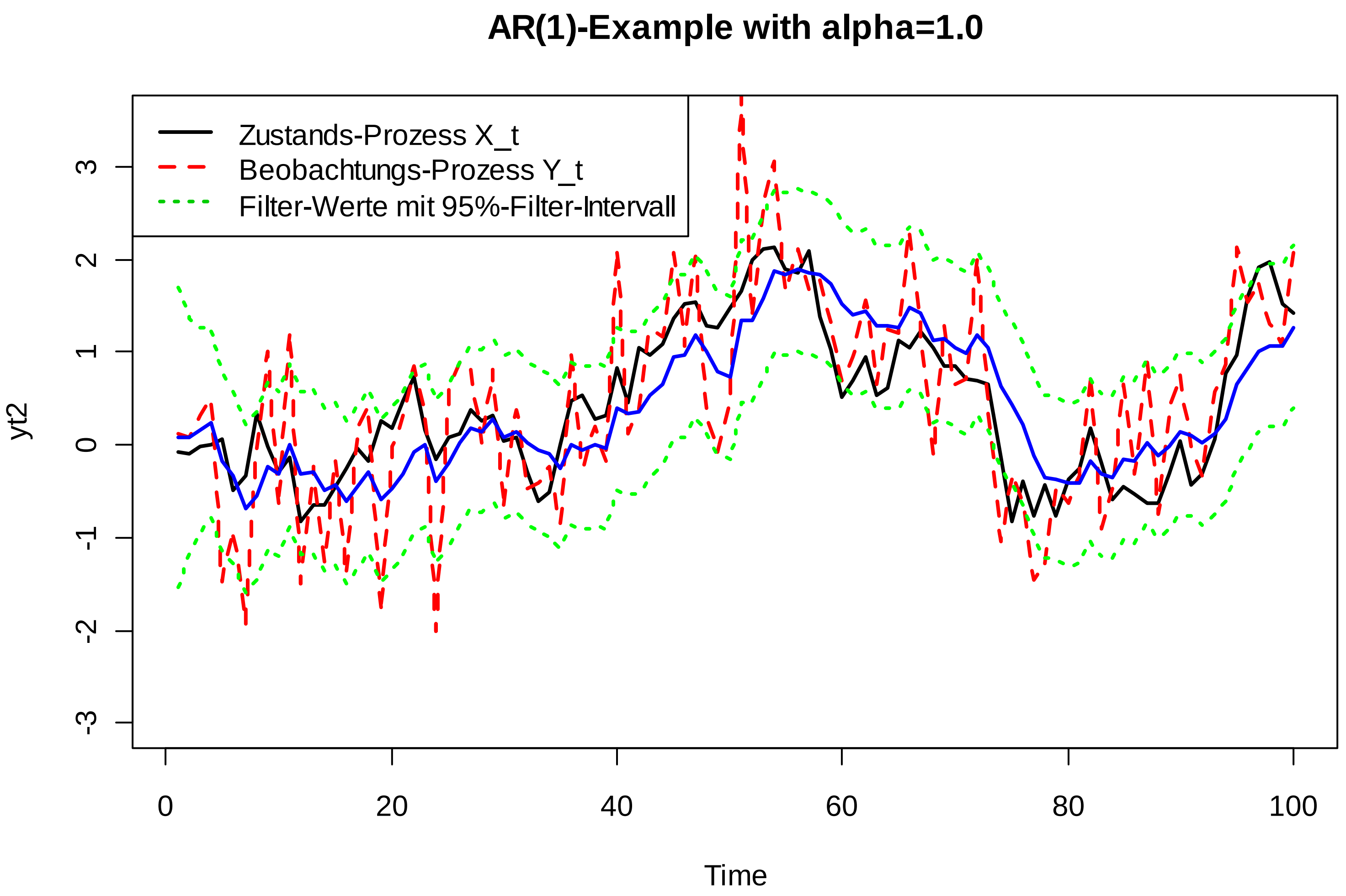
AR(1)-Example with α=1.0



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AR(1)-Smoothing with α=1.0