Applied Time Series Analysis
FS 2011 – Week 08

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ARMA(p,q)-Models

An ARMA(p,q)-model combines AR(p) and MA(q):

$$X_t = \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + E_t + \beta_1 E_{t-1} + \ldots + \beta_q E_{t-q}$$

where E_t are i.i.d. innovations (=a white noise process).

It‘s easier to write an ARMA(p,q) with the characteristic polynom:

$\Phi(B) X_t = \Theta(B) E_t$, where

$\Phi(z) = 1 - \alpha_1 z - \ldots \alpha_p z^p$ is the cP of the AR-part, and

$\Theta(z) = 1 - \beta_1 z - \ldots \beta_q z^q$ is the cP of the MA-part

Properties of ACF/PACF in ARMA(p,q)

	ACF	PACF
AR(p)	exponential decay	cut-off at lag p
MA(q)	cut-off at lag q	exponential decay
ARMA(p,q)	as AR(p) for k>q	as MA(q) for k>p

→ all linear time series processes can be approximated by an ARMA(p,q) with possibly large p,q. They are thus are very rich class of models.

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Fitting ARMA(p,q)

What needs to be done?

1) **Achieve stationarity**
→ transformations, differencing, modeling, …

2) **Choice of the order**
→ determining (p,q)

3) **Parameter estimation**
→ Estimation of $\alpha, \beta, \mu, \sigma_E^2$

4) **Residual analysis**
→ if necessary, repeat 1), and/or 2)-4)

Forecasting with ARMA(p,q)

There are 3 main sources of uncertainty:

1) Does the data generating model from the past also apply in the future?

2) Is the ARMA(p,q)-model we fitted to the data $\{x_1, \ldots, x_n\}$ correctly chosen?

3) Are the parameters α, β, σ_E^2 and μ accurately estimated?

→ **we will here restrict to short-term forecasting!**

How to Forecast?

Probabilistic principle for point forecasts:

$$\hat{X}_{n+k,n} = E\left[X_{n+k} \mid X_1^n \right]$$

→ we forecast the expected value, given our observations

Probabilistic principle for prediction intervals:

$$Var\left(X_{n+k} \mid X_1^n \right)$$

→ we use the conditional variance

How to Apply the Principles?

- The principles provide a nice setup, but are only useful and practicable under additional assumptions.

- Whereas for AR(p), knowing the last p observations is sufficient for coming up with a forecast, ARMA(p,q) models require knowledge about the infinite past.

- In practice, one is using recursive formulae

→ **see blackboard for the derivation in the MA(1) case!**

MA(1) Forecasting: Summary

- We have seen that for an MA(1)-process, the k-step forecast for k>1 is equal to μ.

- In case of k=1, we obtain for the MA(1)-forecast:

$$\hat{X}_{n+1,n} = \mu + \beta_1 \cdot E[E_n \mid X_1^n]$$

The conditional expectation is (too) difficult to compute

- As a trick, we not only condition on observations 1,…,n, but on the infinite past:

$$e_n := E[E_n \mid X_{-\infty}^n]$$

MA(1) Forecasting: Summary

- We then write the MA(1) as an AR(∞) and solve the model equation for E_n :

$$E_n = \sum_{j=0}^{\infty} (-\beta_1)^j \cdot (X_{n-j} - \mu)$$

- In practice, we plug-in the time series observations x_{n-j} where available. For the „early" times, where we don't have observations, we plug-in $\hat{\mu}$.

- This is of course only an approximation to the true MA(1)-forecast, but it works well in practice, because of:

$$|\beta_1| < 1$$

ARMA(p,q) Forecasting

As with MA(1)/MA(q) forecasting, we face problems with

$$E[E_{n+1-j} \mid X_{-\infty}^{n}]$$

which is difficult to compute. We use the same tricks as for MA(1) and obtain

$$\hat{X}_{n+k,n} = \mu + \sum_{i=1}^{p} \alpha_i (E[X_{n+k-i} \mid X_{-\infty}^{n}] - \mu)$$

$$+ E[E_{n+k} \mid X_{-\infty}^{n}] - \sum_{j=1}^{q} \beta_j E[E_{n+k-j} \mid X_{-\infty}^{n}]$$

where …

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ARMA(p,q) Forecasting

…where

$$E[X_t \mid X_{-\infty}^{n}] = \begin{cases} x_t & \text{if } t \le n \\ \hat{X}_{t,n} & \text{if } t > n \end{cases}$$

and

$$E[E_t \mid X_{-\infty}^{n}] = \begin{cases} e_t & \text{if } t \le n \\ 0 & \text{if } t > n \end{cases}$$

with

$$e_t = x_t - \mu - \sum_{i=1}^{p} \alpha_i (x_{t-i} - \mu) + \sum_{j=1}^{q} \beta_j e_{t-j}$$

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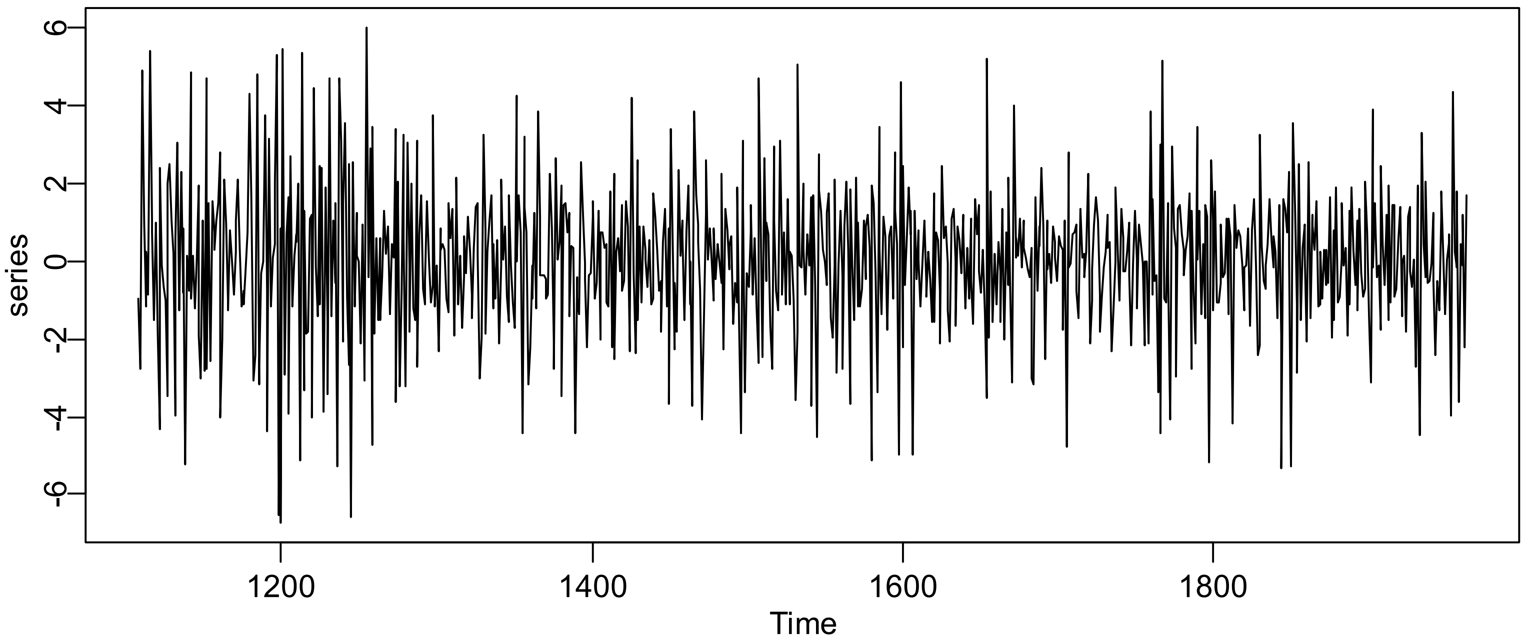
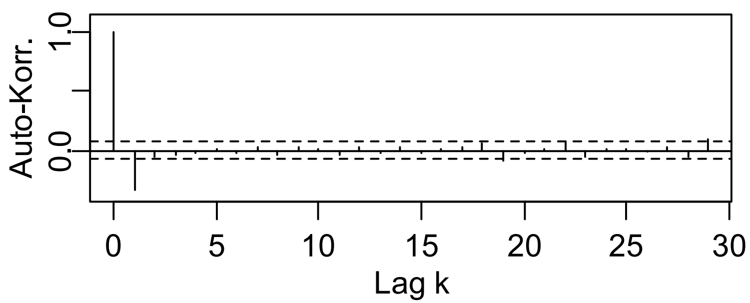
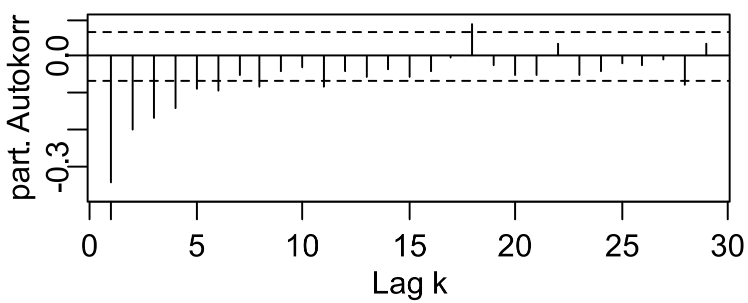
ARMA(p,q) Forecasting: Douglas Fir







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ARMA(p,q) Forecasting: Example