Solutions to Exercise Sheet 2

1. a) The null hypothesis: there are an average of 10 purple Smarties in each packet.
The alternative hypothesis: there are, on average, less than 10 purple Smarties in each packet.

b) Each individual event has a very small probability.
Thus even finding *exactly* 10 purple Smarties in a packet – when the *average* per packet is 10 – is quite an improbable event.
To reject the null hypothesis, we must instead show that the probability (under the null hypothesis) of observing an event *at least as exteme as this* is very small.
Thus we must show that the probability (given the null hypothesis) of finding *at most* five purple Smarties in a packet is small.

c) We assume that the colours of Smarties in a packet are independent of each other. Then X, the number of purple Smarties, follows the binomial distribution

$$X \sim \mathcal{B}\langle n, p\rangle,$$

where p the probability that a randomly chosen Smartie is purple. Under the null hypothesis, this probability is small:

$$p = \frac{\text{average number of purple Smarties in a packet}}{\text{total number of Smarties in a packet}} = \frac{10}{1000} = 1\%.$$

As $n = 1000$ is rather large (and p is small), the distribution of X can be approximated by a Poisson distribution with parameter

$$\lambda = n \cdot p = 10,$$

yielding

$$P[X = k] = e^{-\lambda}\frac{\lambda^k}{k!}.$$

So under the null hypothesis, the probability of finding at most five purple Smarties in a packet is

$$\begin{aligned}
P[X \leq 5] &= P[X = 0] + P[X = 1] + \ldots + P[X = 5] && (1)\\
&= e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} + \ldots + e^{-\lambda}\frac{\lambda^5}{5!} && (2)\\
&= e^{-\lambda}\sum_{k=0}^{5}\frac{\lambda^k}{k!} && (3)\\
&= 0.067 = 6.7\%. && (4)
\end{aligned}$$

NB: We also could have computed the exact probability using the binomial distribution (this was not required here, though). This yields the probability 6.61%. Thus the Poisson approximmation is not bad.

d) The probability we obtained is still a little bit too large to disprove the manufacturer's claim. Usually we ask for this probability (the p-value) to be less than 5% at the very least.

If however we had only found four purple Smarties in the packet, the manufacturer's claim could have beeen rejected (at the 5% level).

2. As the sample was taken *randomly* from a *large* batch, we can assume that the number of deficient test tubes in the sample – which we shall denote by X – follows a binomial distribution (cf. Problem 1 of Problem Sheet 2). The manufacturer would like to show that the proportion of deficient items in the batch is less than 10%. The null hypothesis chosen will be that 10% of the batch is deficient (the manufacturer would like to reject the null hypothesis).

a) $H_0 : X \sim \mathcal{B}\langle n, \pi_0 \rangle$, where $n = 50$ and $\pi_0 = 0.1$

b) $H_A : X \sim \mathcal{B}\langle n, \pi \rangle$, where $n = 50$ and $\pi < 0.1$.

c) The rejection region for a one-sided test at significance level 5% can be computed quite simply in our example. The critical value satisfies the following condition:

$$P_0\langle X \leq c \rangle = \sum_{k=0}^{c} \binom{n}{k} \pi_0^k (1-\pi_0)^{n-k} \leq 0.05$$

We find that:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
P_0\langle X = 0 \rangle & = & 0.0052 & P_0\langle X \leq 0 \rangle & = & 0.0052 \\
P_0\langle X = 1 \rangle & = & 0.0286 & P_0\langle X \leq 1 \rangle & = & 0.0338 \\
P_0\langle X = 2 \rangle & = & 0.0779 & P_0\langle X \leq 2 \rangle & = & 0.1117
\end{array}$$

The rejection region K at the significance level 5% is $K = \{X \leq 1\}$. Thus the acceptance region is $K^c = \{X > 1\}$.

d) The observed event was that $X = 3$ deficient test tubes were among the batch. This observation lies in the acceptance region. We say that the null hypothesis is **kept**, i.e. it is plausible that the whole batch contains 10% (or more) deficient test tubes, even though only 6% were found in the samplle. The observed event is not "extreme" (i.e. improbable) enough to justify the rejection of the null hypothesis.