Solutions to Exercise Sheet 1

1. a) Assume one of the dice is red and other other one black. If both the red and the black die land as a 1, the event “two 1s” occurs. The event “a 1 and a 2” can occur in two different ways: as a red 1 and black 2, or as a redd 2 and black 1. We denote these events using parentheses, writing first the result of the red die and then that of the black die. Thus the above events are: $(1,1), (1,2)$ and $(2,1)$. This leads to the following event space:

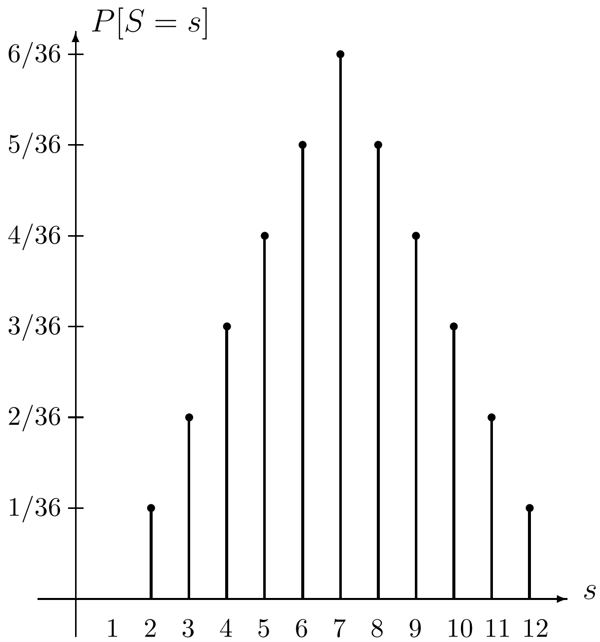
$$\Omega = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(2,6),(3,1)\ldots,(6,6)\}$$

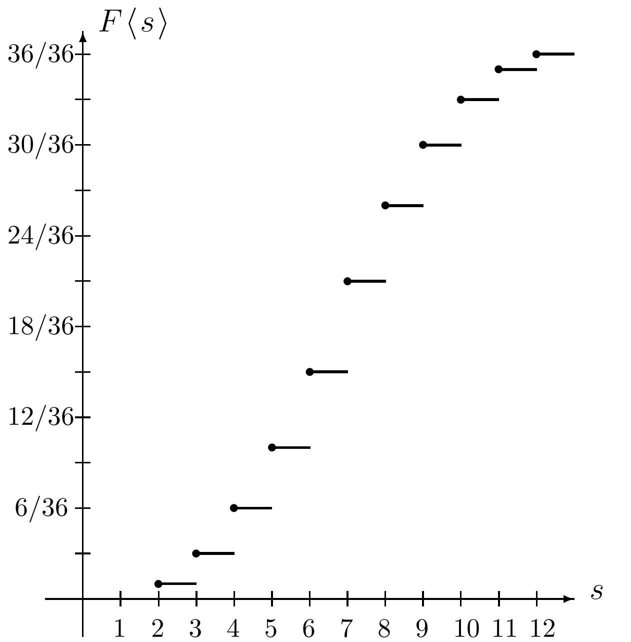
The event space Ω consists of 36 elementary events.

b) The random variable S is the sum of eyes on the dice. Its range is $W_S = \{2,3,4,\ldots,11,12\}$. The following table describes its probability distribuution:

$S = s$	2	3	4	5	6	7	8	9	10	11	12
$P\langle S = s\rangle$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c) Probability distribution and (cumulative) distribution function





2. a) Let X denote the number of lesser-quality test tubes in the sample. Assuming independence and equal manufacturing conditions for all items (each item thus incurring the same risk of loss of quality), X follows a binomial distribution.

b) Under the conditions given, we have $X \sim \text{Bin}(n, \pi)$, where $n = 50$ and $\pi = 0.1$. The probability we seek is then

$$\text{P}[X = 3] = \binom{50}{3} 0.1^3 \cdot 0.9^{47} = 0.139.$$

c) If the batch contains 10% of lower-quality test tubes, then X follows the binomial distribution $\text{Bin}(n, \pi)$, where $n = 50$ and $\pi = 0.1$. We then have

$$\text{P}[X \leq 3] = \sum_{k=0}^{3} \binom{50}{k} (0.1)^k \cdot (0.9)^{50-k} = 0.25.$$

d) The problem faced by the manufacturer is to set a upper bound for the number of deficient items in a batch. If the batch is of bad quality, this upper bound should be exceeded with high probability. If the batch quality is acceptable, though, this bound should not be exceeded often.

Note:

Problem c) illustrates how chance can be an obstacle. There is a 25% probability of randomly obtaining no more than three deficient items in a sample (of $n = 50$) out of a batch containing 10% deficient items. However, 3 deficient items are a mere 6% of this sample.

This shows that even when a sample of $n = 50$ items has deficiencies in no more than 6% of the items, we cannot necessarily conclude that the whole batch contains less than 10% of deficient items. Such a sample will be drawn – and the wrong conclusion reached – in an average of a quarter of cases! The question now is: how small must the proportion of deficient test tubes be in the sample, to guarantee (with resonable certainty) that the whole batch has a sufficient quality. This leads to a statistical testing problem.

3. a) Let X be the number of individually contaminated samples in a pooled sample. The probability p of a sample being contaminated is 0.02. Assuming independence of the samples, we have $X \sim \text{Bin}(n = 10, p = 0.02)$.

Thus the probabilityfor the pooled sample to be free of any contamination is given by

$$\text{P}[X = 0] = \binom{10}{0} 0.02^0 \cdot 0.98^{10} = 0.98^{10} = 0.817\,.$$

Alternative solution: independently of all others, each sample has a 98% probability of being clean. Thus by the **(multipllicative law for independent events)** we have

$$\text{P}[\text{all samples clean}] = \prod_{i=1}^{10} \text{P}[\text{i-th sample clean}] = 0.98^{10} = 0.817\,.$$

b) The random variable Y only takes on the values 1 and 11, as:

- if all samples are clean, one investigation suffices: $Y{=}1$
- if not, each sample must be examined individually (even if one contaminated sample is found, the rest must still be examined, as several could be contaminated at the same time!); thus $Y{=}11$.

Consequently,

$$\begin{cases} \mathrm{P}[Y=1] = & \mathrm{P}[\text{all samples are clean}] = & 0.817 \\ \mathrm{P}[Y=11] = & 1-\mathrm{P}[Y=1] = & 0.183 \end{cases}$$

c) The average number of analyses per pooled sample is the mean of the random variable Y:

$$\begin{aligned} \mathbf{E}[Y] &= \sum_{k=0}^{\infty} k\,\mathrm{P}[Y=k] = 1\cdot\mathrm{P}[Y=1] + 11\cdot\mathrm{P}[Y=11] \\ &= 1\cdot 0.817 + 11\cdot 0.183 = 2.83 \end{aligned}$$

The „average“ saving is $10 - 2.83 = 7.17 \approx 7$ analyses.